MICROWAVES

BY

Prof.Dr. FİLİZ GÜNEŞ

YILDIZ TECHNICAL UNIVERSITY
ELECTRIC AND ELECTRONIC FACULTY
ELECTRONICS AND COMMUNICATIONS ENGINEERING,BEŞİKTAŞ-İSTANBUL

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SUBJECTS

Part 1 : Transmission Line Theory
Part 2 : Impedance Transformation And Matching
Part 3 : Rectangular And Circular Waveguides

LOW-FREQUENCY ELECTRIC CIRCUITS AND TRANSMISSION LINES

Transmission lines differ from the low-frequency electric circuits in the following features :

- Maximum physical dimension of a low-frequency electric circuit is very much smaller than the operation wavelength , so the propagation time for an electric signal is so short that it does not need to be taken into account;

- Transmission lines are usually a considerable multiples of wavelength and may even be many wavelengths long , so PROPAGATION TIME for the electric signal along the line has to be taken into account;

- The elements in a low- frequency electric circuit can be described by lumped parameters so that currents flowing in lumped circuit elements do not vary specially along the elements,and no standing waves exist ⇔LUMPED-PARAMETER CIRCUIT;

- A transmission line,on the other hand can be considered as a DISTRIBUTED-PARAMETER CIRCUIT which can be described by the circuit parameters distributed throughout its length.Except matched conditions,STANDING WAVES exist in a transmission line.In otherwords, voltages and currents can vary in magnitude and phase over the length of the transmission line ⇔DISTRIBUTED - PARAMETER CIRCUIT;

VOLTAGE AND CURRENT ON A TRANSMISSION-LINE

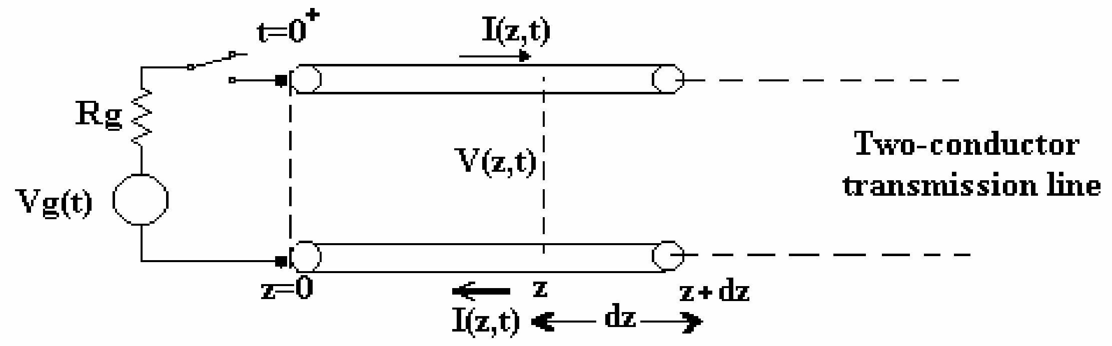


Figure-1

Equivalent circuits of differential length dz s of the two-conductor lossy and lossless transmission lines can be given by the circuits in the Fig-2 and 3,respectively.

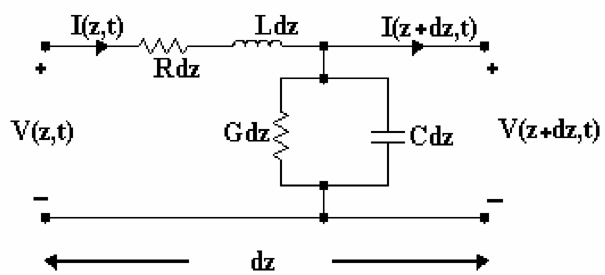


Figure-2:Equivalent circuit for a lossy transmission line in the differential length

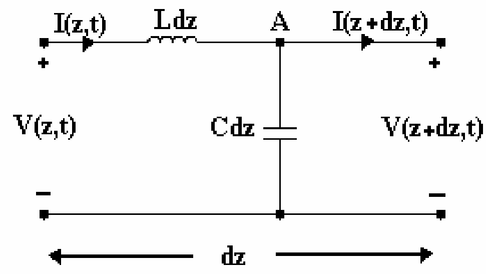


Fig-3 :Equivalent circuit for an ideal transmission line in the differential length

A transmission line, in a differential length dz , can be described by the following four parameters;

R , resistance per unit length in Ω/m
L , inductance per unit length in H/m
G ,conductance per unit length in S/m
C ,capacitance per unit length in F/m

where R and L are the series elements, G and C are the shunt elements.R and G equals to zero in an ideal (lossless) transmission line as shown in **Fig. 3**.

If the quantities V(z,t) and V(z+dz,t) denote the instantenous voltages at z and z+dz positions of the line respectively ; the relation betweeen these instantenous voltages can be given as follows:

$$V(z+dz,t) = V(z,t) + \frac{\partial V}{\partial z} dz \tag{1}$$

Similary , if the quantities I(z,t) and I(z+dz,t) denote the instantenous currents at z and z+dz respectively ; the relation betweeen these instantenous currents can be expressed as follows:

$$I(z+dz) = I(z,t) + \frac{\partial I}{\partial z} dz \tag{2}$$

Applying the Kirschhoff's voltage law to the equivalent circuit of the ideal (lossless) transmission line in Fig.3, we obtain ;

$$-V(z,t) + Ldz\frac{\partial I(z,t)}{\partial t} + V(z+dz,t) = 0 \tag{3}$$

which leads to;

$$-\frac{V(z+dz,t)-V(z,t)}{dz} = L\frac{\partial I(z,t)}{\partial t} \tag{4}$$

in the limit $\Delta z \to 0$, equation(4) becomes

$$-\frac{\partial V(z,t)}{\partial z} = L\frac{\partial I(z,t)}{\partial t} \tag{5}$$

Similarly, applying theKirschhoff's current law to the ideal transmission line in Figure-3,we have:

$$-I(z,t) + Cdz\frac{\partial V(z+dz,t)}{\partial t} + I(z+dz,t) = 0 \qquad (6)$$

Dividing dz and letting dz approach zero,equation (6) becomes,

$$\frac{-\partial I(z,t)}{\partial z} = C\frac{\partial V(z,t)}{\partial t} \qquad (7)$$

So equations (6) and (7) give the relations between voltage and current at the instant t and on location z of an ideal transmision line.

VOLTAGE AND CURRENT WAVES ON AN IDEAL TRANSMISSION LINES

If the partial derivatives of the equations (5) and (7) with respect to the time and space are taken respectively and then combined together , one obtains **ONE DIMENSIONAL CURRENT WAVE EQUATION:**

$$\frac{\partial^2 V(z,t)}{\partial z \partial t} = -L\frac{\partial^2 I(z,t)}{\partial t^2} \qquad (8)$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = -C\frac{\partial^2 V(z,t)}{\partial t \partial z} \qquad (9)$$

$$LC\frac{\partial^2 I(z,t)}{\partial t^2} = \frac{\partial^2 I(z,t)}{\partial z^2} \Rightarrow \frac{\partial^2 I(z,t)}{\partial z^2} - LC\frac{\partial^2 I(z,t)}{\partial t^2} = 0 \qquad (10)$$

Since similiar process can be repeated for the voltage on the transmission line, so one can define **ONE DIMENSIONAL WAVE OPERATOR:**

$$\left\{\frac{\partial^2}{\partial z^2} - LC\frac{\partial^2}{\partial z^2}\right\}\begin{matrix} V(z,t) \\ \\ I(z,t)\end{matrix} = 0 \quad \textbf{One dimensional wave equation} \qquad (11)$$

where $v \overset{\Delta}{=} \frac{1}{\sqrt{LC}}$ m/s is the phase velocity. **(**12)

If the u(z,t) denotes the solution of the one dimensional wave equation in (11) which can be either the voltage or current wave , so general expression of the u(z,t) can be given as follows:

$$u(z,t) = \underbrace{u^{+}\left(t - \frac{z}{v}\right)}_{\substack{\textbf{wave component in} \\ \textbf{+ z-direction}}} + \underbrace{u^{-}\left(t + \frac{z}{v}\right)}_{\substack{\textbf{wave component in} \\ \textbf{- z-direction}}} \qquad (13)$$

So the voltage waves can be expressed as follows:

$$V(z,t) = V^{+} f^{+}\left(t - \frac{z}{v}\right) + V^{-} f^{-}\left(t + \frac{z}{v}\right) \qquad (14)$$

In equation (14) , V^{+} and V^{-} denote amplitudes of the voltage waves propagating with the phase velocity v in +z and –z directions, respectively.
Substituting (14) in the relations given by (5) and (7) , one can write the following expressions for the current waves:

$$I(z,t) = I^{+} f^{+}(t - \frac{z}{v}) - I^{-} f^{-}(t + \frac{z}{v}) \qquad (15)$$

$$I^{+} = v.C.V^{+}$$

$$I^{-} = v.C.V^{-} \qquad (16)$$

By using equations given by (16) **,** we have

$$Z_C \;\hat{=}\; \frac{V^{+}}{I^{+}} = \frac{V^{-}}{I^{-}} = (v \cdot C)^{-1} = \left(\frac{1}{LC} \cdot C\right)^{-1} = \sqrt{\frac{L}{C}} \;\; \Omega \qquad (17)$$

Z_C is called the characteristic impedance of the line.

Using the equation (17) , we can express current waves in terms of the voltage waves:

$$I(z,t) = \frac{V^{+}}{Z_c} f^{+}(t - \frac{z}{v}) - \frac{V^{-}}{Z_c} f^{-}(t + \frac{z}{v}) \qquad (18)$$

VOLTAGE AND CURRENT WAVES ON A SEMI-INFINITE LOSSLESS TRANSMISSION LINE

If there is no reflection wave,such as in the case of the semi-infinite transmission line ,so the voltage and current waves have only single component propagating in +z direction :

$$V(z,t) = V^+ f^+\left(t - \frac{z}{v}\right)$$

$$I(z,t) = \frac{V^+}{Z_c} f^+\left(t - \frac{z}{v}\right) \qquad (19)$$

If we apply the Kirschhoff 's voltage law at the location of the z=0 , we have

$$V_g(t) = V(0, t) + I(0, t) \cdot R_g \qquad (20)$$

Using

$$f^+(t) \stackrel{\Delta}{=} f^+(0, t) \qquad (21)$$

Then the expression (20) becomes

$$V_g(t) = V^+ f^+(t) + \frac{V^+}{Z_C} R_g f^+(t) \qquad (22\ )$$

So $V^+ f^+(t)$ is obtained from the equation (22) as

$$V^+ f^+(t) = \frac{Z_C}{Z_C + R_g} V_g(t) \qquad (23)$$

So the voltage and current expressions at the t instant and on the z location of the transmission line can be given as

$$V(z, t) = \frac{Z_C}{Z_C + R_g} V_g\left(t - \frac{z}{v}\right)$$

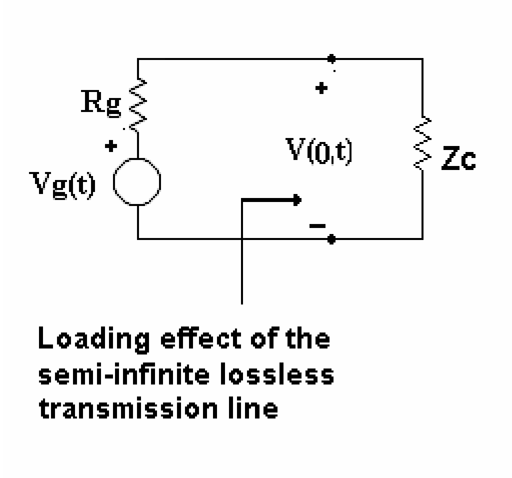
(24)

$$I(z,t) = \frac{1}{Z_C + R_g} V_g\left(t - \frac{z}{v}\right) \tag{25}$$

where $\quad v \triangleq \dfrac{1}{\sqrt{LC}} \quad$ and $\quad Z_C \triangleq \sqrt{\dfrac{L}{C}}$.

Using the equation (23), the equivalent circuit of a semi-infinite transmision line at the z=0 location can be given as shown in **Figure 4.**

Figure-4



The input impedance of a semi-infinite transmission line at the z=0 position , is equal to the characteristic impedance of the line **:**

$$Z_{in} = Z_C \tag{26}$$

TERMINATED LINE : RESISTIVE TERMINATION

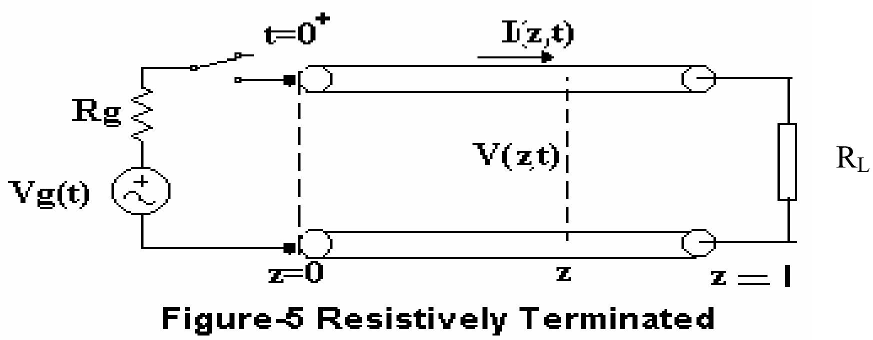


Figure-5 Resistively Terminated Transmission Line

Boundary conditions at the z=0 location are as follows:

$$V(\ell,t) = V_L = I_L R_L \tag{27}$$

$$I(\ell,t) = I_L \tag{28}$$

(1) If $R_L = Z_c$, $V(\ell,t) = V_L = I(\ell,t)Z_C$ is satisfied by only $V^+(\ell,t)$ and $\mathbf{I^+(\ell,t)}$ waves $\Rightarrow$ The energy carried by the incident wave is completely absorbed by the load . $\Rightarrow$ $V^-(\ell,t) = 0$

(2) $\mathbf{R_L \neq Z_C} \Rightarrow$ in order that the boundary conditions given by (27) and (28) to be satisfied the reflected wave components have to exist. Now the generation of the reflected wave will be formulated in terms of the source voltage waveform. Firstly the voltage wave incident across the load can be expressed in terms of the the source voltage waveform as

$$\upsilon_i(\ell,t) = \frac{Z_C}{Z_C + R_g}\upsilon_g\left(t - \frac{\ell}{\upsilon}\right) = V^+\upsilon_g\left(t - \frac{\ell}{\upsilon}\right) \tag{29}$$

and the current through the load is given by

$$I_i(\ell,t) = \frac{1}{Z_C}\upsilon_i(\ell,t) \tag{30}$$

Since there exists the only single wave component until the waves come to the load , so we can express the reflected voltage wave component as

$$\upsilon_r(z,t) = V^-\upsilon_g\left(t - \frac{\ell}{\upsilon} - \frac{\ell - z}{\upsilon}\right)$$

$$\upsilon_r(z,t) = V^-\upsilon_g\left(t + \frac{z}{\upsilon} - \frac{2\ell}{\upsilon}\right) \tag{31}$$

The reflected voltage and current waves of the load can be given as

$$\upsilon_r(\ell, t) = V^{-}\upsilon_g\left(t - \frac{\ell}{\upsilon}\right) \quad \mathbf{I}_r(\ell, \mathbf{t}) = \frac{\mathbf{V}^{-}}{\mathbf{Z}_C}\upsilon_g\left(\mathbf{t} - \frac{\ell}{\upsilon}\right)$$

(32)

The boundary condition is $\upsilon_L = \upsilon(\ell, \mathbf{t}) = \mathbf{I}(\ell, \mathbf{t})\mathbf{R}_L$ and substituting the incident and reflected wave expressions into the boundary condition, we have

$$\upsilon_L(z, t) = \frac{1}{Z_C}(V^{+} - V^{-})\upsilon_g\left(t - \frac{\ell}{\upsilon}\right)R_L$$

$$\upsilon_L(z, t) = (V^{+} + V^{-})\upsilon_g\left(t - \frac{\ell}{\upsilon}\right) \qquad \textbf{(33)}$$

So using (33) , the reflection coefficient can be defined as

$$\Gamma_g \overset{\Delta}{=} \frac{V^{-}}{V^{+}} = \frac{R_L - Z_C}{R_L + Z_C} \qquad \textbf{(34)}$$

Properties :

- $\Gamma_g \leq 1$
- $R_L = Z_C \Leftrightarrow V^{+} = 0$ (Termination by the Characteristic imp.)
- $R_L = 0 \Leftrightarrow \Gamma_L = -1 \Leftrightarrow V = -V^{+}$ (Short-circuit Termination)
- $R_L \Rightarrow \infty \Leftrightarrow \Gamma_L = 1 \Leftrightarrow V^{-} = V^{+}$ (Open-circuit Termination)

ν(z,t) , i(z,t) at the location of z=0 should satisfy the boundary condition

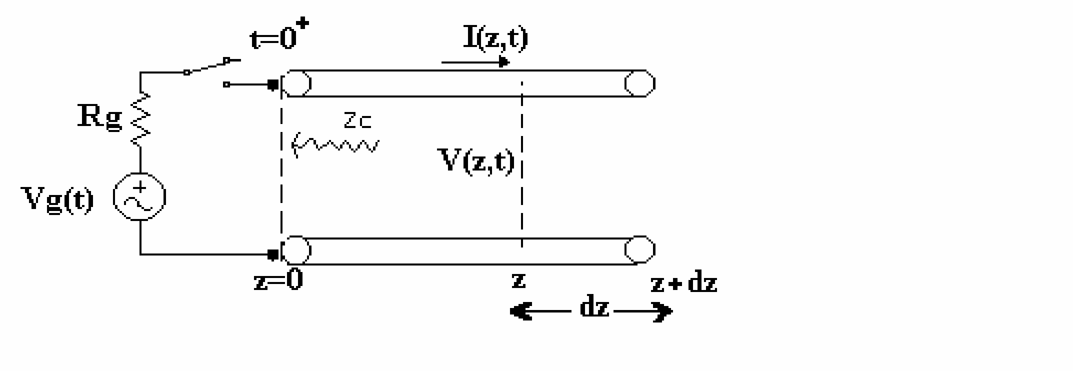


Figure 6

If we make $v_g(t) = 0$, at $z = 0$ and applying the Kirschhoff Voltage and Current Laws, we obtain $\Gamma_g = \frac{R_g - Z_C}{R_g + Z_C}$. So, for $R_g \neq Z_C$, there is a reflected wave travelling towards the load too.

MULTIPLE - REFLECTION THEORY

According to this theory, $\upsilon(z,t)$ can be expressed as the convergent series of the incident wave and its resulted reflected wave components:

$$\begin{aligned}
\upsilon(z,t) = {} & V^+\upsilon_g\left(t-\frac{z}{\upsilon}\right)+\Gamma_L{}^2V^+\upsilon_g\left(t+\frac{z}{\upsilon}-\frac{2\ell}{\upsilon}\right)U(t-\frac{\ell}{\upsilon})+ \\
& \Gamma_g\Gamma_L V^+\upsilon_g\left(t-\frac{z}{\upsilon}-\frac{2\ell}{\upsilon}\right)U(t-\frac{2\ell}{\upsilon})+ \\
& \Gamma_g\Gamma_L{}^2V^+\upsilon_g\left(t+\frac{z}{\upsilon}-\frac{4\ell}{\upsilon}\right)U(t-\frac{3\ell}{\upsilon})+ \\
& \Gamma_g{}^2\Gamma_L{}^2V^+\upsilon_g\left(t-\frac{z}{\upsilon}-\frac{4\ell}{\upsilon}\right)U(t-\frac{4\ell}{\upsilon})+....
\end{aligned} \tag{35}$$

where $U(t-\frac{z}{\upsilon})$ is the Unit Step function which can be expressed as

$$U(t-\frac{z}{\upsilon})=\begin{cases}1\rightarrow t>\dfrac{z}{\upsilon}\\ 0\rightarrow \text{otherwise}\end{cases} \tag{36}$$

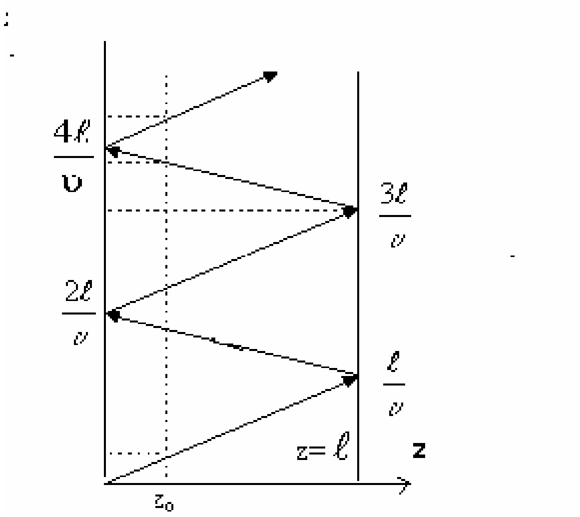


Figure 7

Γ_L=0.5 Γ_g= - 0.5

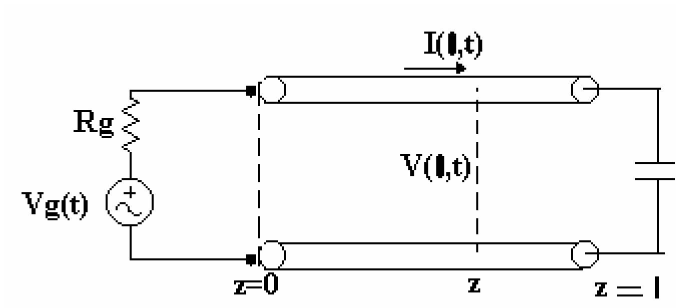


Figure 8

Capacitive Termination

Definition equations of the termination can be written as:

$$Q_c(t)=C.V_c(t) \quad \text{or} \quad I_c(t)=C\frac{dV_c(t)}{dt} \tag{37}$$

REFLECTION DIAGRAMS

The preceding step-by-step construction and calculation procedure of the voltage and current at a particular time and location on a transmission line with an arbitrary resistive termination tends to be tedious and difficult to visualize since one has to consider so many reflected waves. In such cases the graphical construction of a reflection diagram will be very helpful. Firstly let us construct ***a voltage reflection diagram.*** A reflection diagram plots the time elapsed after a change in circuit conditions versus the distance z from the source end.
The voltage reflection diagram of the circuit in the Fig.1 is given in the Fig.2.

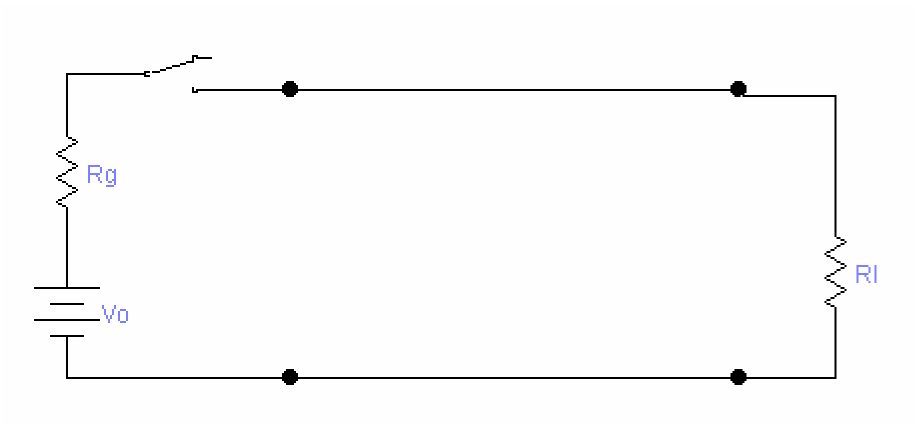


Figure-1

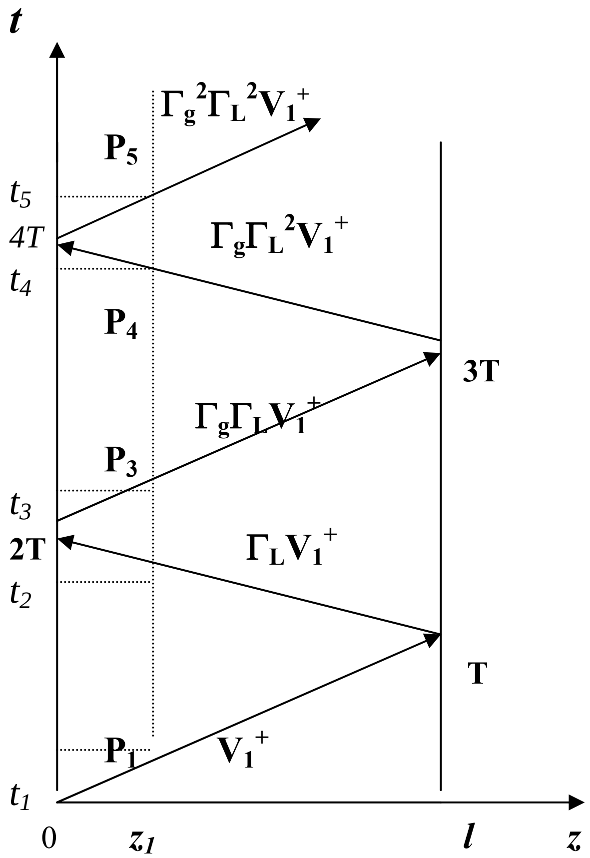


Figure-2

It starts with a wave V_1^+ at $t=0$ travelling from the source end (z=0) in the +z direction with a velocity u. This wave is represented by the directed straight line marked V_1^+ from the origin. This line has a positive slope equal to $1/u$. When the V_1^+ wave reaches the load at $z=l$, a reflected wave $V_1^- = \Gamma_L V_1^+$ is created if $R_L \neq R_0$. The V_1^- wave travels in the $-z$ direction and is represented by the directed line marked $\Gamma_L V_1^+$ with a negative slope equal to $-1/u$.

The V_1^- wave returns to the source end at instant $t=2T$ and gives rise to another reflected wave $V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$, which is represented by a second directed line with a positive slope. This process continues back and forth infinitely. The voltage reflection diagram can be used conveniently to determine the voltage distribution along the transmission line at a given time as well as the variation of the voltage as a function of time at an arbitrary point on the line.

The voltage distribution along the line at ***t=t₄* (*3T<t₄<4T*).**

1. Mark t_4 on the vertical t-axis of the voltage reflection diagram.

2. Draw a horizontal line from t_4, intersecting the directed line marked $\mathbf{\Gamma_g\Gamma_L^2V_1^+}$ at P_4.(All directed lines above P_4 are irrelevant to our problem because they pertain to $t>t_4$.)

3. Draw a vertical line through P_4, intersecting the horizontal z-axis at z_1. In the range of $0<z<z_1$, the voltage has a value equal to $\mathbf{V_1=V_1^+(1+\Gamma_L+\Gamma_g\Gamma_L)}$; and in the range of $z_1<z<l$ the voltage is equal to $\mathbf{V_1^++V_1^-+V_2^++V_2^-}$ $\mathbf{=V_1^+(1+\Gamma_L+\Gamma_g\Gamma_L+\Gamma_g\Gamma_L^2)}$. So there is a voltage discontinuity equal to $\mathbf{\Gamma_g\Gamma_L^2V_1^+}$ at $z=z_1$ position.

4. The voltage distribution along the line at $t=t_4$, $V(z,t_4)$, is then as shown in that diagram plotted for $R_L=3R_0 \Leftrightarrow \Gamma_L=1/2$ and $R_g=2R_0$ $\Leftrightarrow$ $\Gamma_g=1/3$.

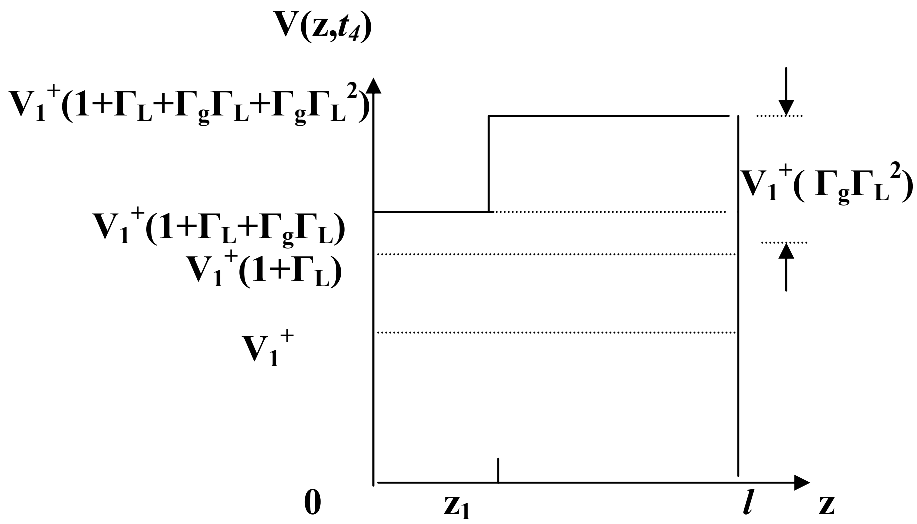


Figure-3

finding the variation of the voltage as the function of time at the point $z=z_1$.

1. Draw a vertical line at z_1, intersecting the directed lines at points P_1, P_2, P_3, P_4, P_5, and so on.(There would be an infinite number of such intersection points if $R_L \neq R_0$ and $R_g \neq R_0$,as there would be an infinite number of directed lines if $\Gamma_L \neq 0$ and $\Gamma_g \neq 0$)

2. From these intersection points, draw horizontal lines intersecting vertical *t*-axis at t_1, t_2, t_3, t_4, t_5 and so on. These are the instants at which a new voltage wave arrives and abruptly changes the voltage at $z=z_1$.

3. The graph of $V(z_1,t)$ is plotted in this diagram for $\Gamma_L=1/2$ and $\Gamma_g=1/3$. When *t* goes to the infinity , the voltage at z_1 (and at all other points along the lossless line) will assume the value $3V_0/5$, as given in equation:

$$\mathbf{V=V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \ldots\ldots\ldots}$$

$$=V_1^+(1+\Gamma_L+\Gamma_g\Gamma_L+\Gamma_g\Gamma_L^2+\Gamma_g^2\Gamma_L^2+\Gamma_g^2\Gamma_L^3+\ldots..)$$

$$=V_1^+[(1+\Gamma_g\Gamma_L+\Gamma_g^2\Gamma_L^2+\ldots.)+\Gamma_L(1+\Gamma_g\Gamma_L+\Gamma_g^2\Gamma_L^2+\ldots..)]$$

$$=V_1^+\left[\left(\frac{1}{1-\Gamma_g\Gamma_L}\right)+\left(\frac{\Gamma_g}{1-\Gamma_g\Gamma_L}\right)\right]$$

$$=V_1^+\left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right)$$

Similar to the voltage reflection diagram in figure-2 a current reflection diagram for the transmission line circuit of figure-1 can be constructed. This is shown in figure-4.

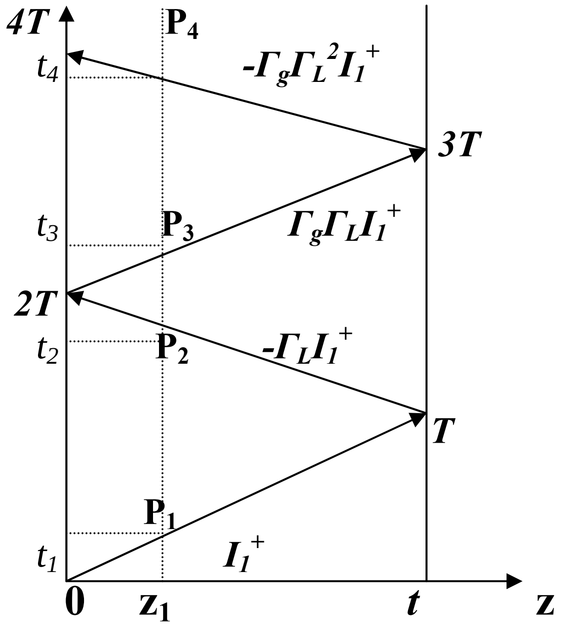


Figure-4

Here we draw directed lines representing current waves . The essential difference between the voltage and current reflection diagrams is in the negative sign associated with the current waves traveling in the –z direction on account of this equation:

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0$$

The current reflection diagram can be used to determine the current distribution along the transmission line at a given time as well as the variation of the current as a function of time at a particular point on the line, following the same procedures outline previously for voltage.

For example we can determine the current at $z=z_1$ by drawing a vertical line z_1 in figure-4,intersecting the directed lines at points P_1,P_2,P_3,P_4 and so on, and by finding the corresponding times t_1,t_2,t_3,t_4 , and so on, as before. Figure-5 as a plot of $I(z_1,t)$versus t, which accompanies the $V(z_1,t)$ graph in figure-6.

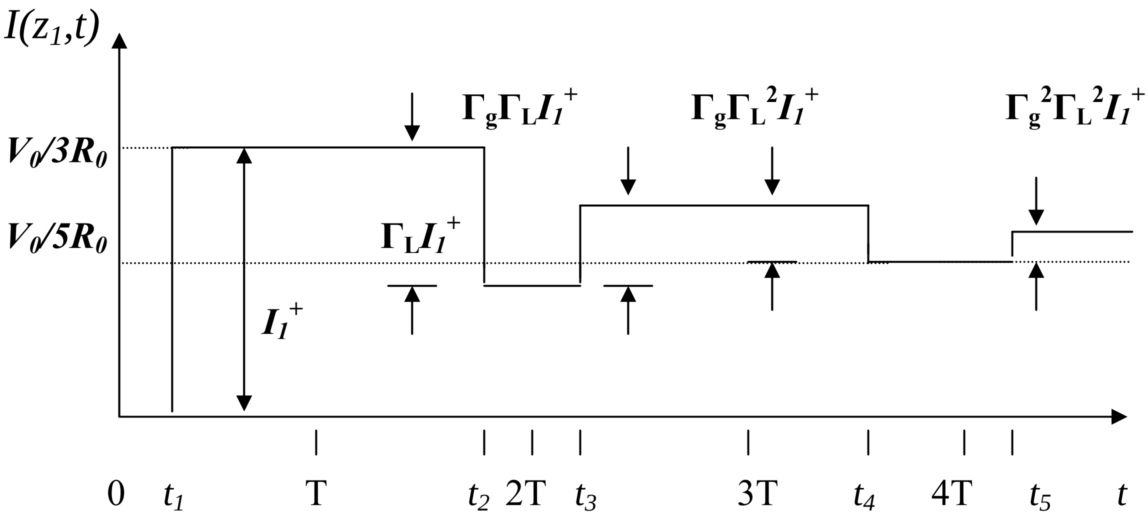


Figure-5

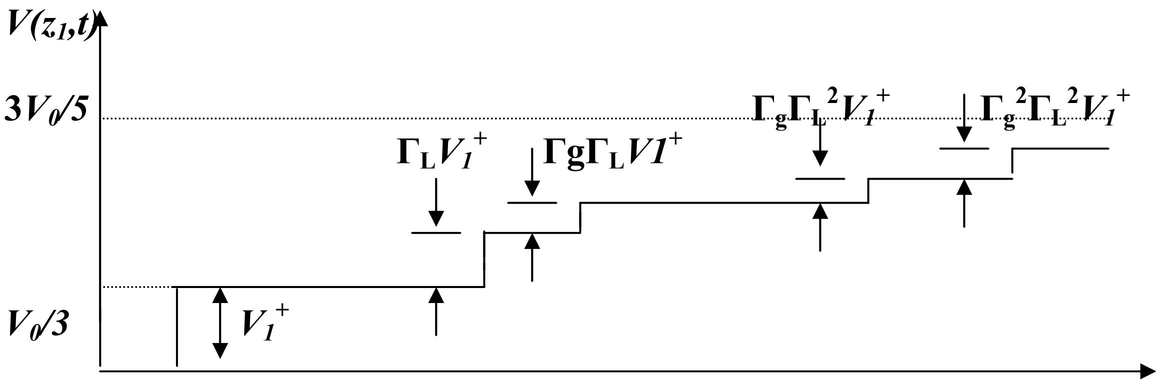


Figure-6

We see that they are quite dissimilar. The current along the line oscillates around the steady-state value of $V_0/5R_0$ as seen at equation:

$$I_L = \left(\frac{1-\Gamma_L}{1-\Gamma_g \Gamma_L} \right) \frac{V_1^+}{R_0}$$

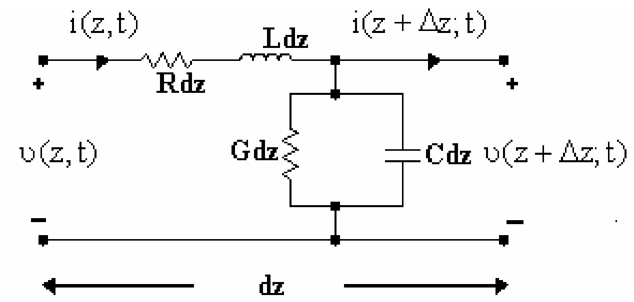
$$I_L = \frac{3}{5}\left(\frac{V_1^+}{R_0}\right) = \frac{V_0}{5R_0}$$

with successively smaller discontinuous jumps at t_1, t_2, t_3, t_4, *etc*. There are two special cases.

1. When $R_L=R_0$ (matched load, $\Gamma_L=0$), the voltage and current reflection diagrams will each have only a single directed line, existing in the interval $0<t<T$, irrespective of what R_g is.

2. When $R_g=R_0$ (matched source $\Gamma_g=0$) and $R_L \neq R_0$, the voltage and current reflection diagrams will each have only two directed lines, existing in the intervals $0<t<T$ and $T<t<2T$.
   In both cases the determination of the transient behavior on the transmission line is much simplified.

LOSSY TRANSMISSION LINES



Relations between V(z,t) ,V(z+dz,t) and I(z,t), I(z+dz,t) can be written as follows:

$$\begin{aligned} i(z+\Delta z;t) &= i(z,t) + \frac{\partial i}{\partial z}dz, \\ \upsilon(z+\Delta z;t) &= \upsilon(z,t) + \frac{\partial \upsilon}{\partial z}dz \end{aligned} \qquad (38)$$

The Kirschhoff's voltage law can be applied to the equivalent circuit of the Lossy transmission line :

$$\upsilon(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - \upsilon(z+\Delta z,t) = 0$$

By dividing with Δz, we have

$$-\frac{\upsilon(z+\Delta z,t) - \upsilon(z,t)}{\Delta z} = R\Delta z i(z,t) + L\frac{\partial i(z,t)}{\partial t}$$

When $\Delta z \to o$ we obtain the derivatives of the voltage and currnent functions,

$$-\frac{\partial \upsilon(z;t)}{\partial z} = Ri(z;t) + L\frac{\partial i(z;t)}{\partial t} \qquad (39)$$

In similiar manner , from the application of the Kirschhoff curent law , we have;

$$i(z;t) - G\Delta z \upsilon(z+\Delta z;t) - C\Delta z \frac{\partial \upsilon(z+\Delta z;t)}{\partial t} - i(z+\Delta z;t) = 0$$

If we divide by Δz , when $\Delta z \to 0$ the limit is

$$\frac{\partial i(z;t)}{\partial z} = G\upsilon(z;t) + C\frac{\partial \upsilon(z;t)}{\partial t} \qquad (40)$$

(39) and (40) are the general transmission line equations $\Leftrightarrow$ **Telegrapher's Equations .**

CONTINUOUS SINUSOIDAL CASE

For the harmonic – time variation , the partial differential equations in (39) and (40) become the ordinary differential equations:

$$\upsilon(z;t) \overset{\Delta}{=} \begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left|\left[V(z)e^{j\omega t}\right]\right. \tag{41}$$

$$i(z;t) \overset{\Delta}{=} \begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left|\left[I(z)e^{j\omega t}\right]\right. \tag{42}$$

If we substitute (41) and (42) into (39) and (40)

$$\frac{dV(z)}{dz} = (R + j\omega L)I(z) \tag{43}$$

$$\frac{dI(z)}{dz} = (G + j\omega C)V(z) \tag{44}$$

From (43) and (44) , we have one - dimensional wave equation for both the voltage and current:

$$\underbrace{\left[\frac{d^2}{dz^2} - \gamma^2\right]}_{\text{one- dimensional wave operator}} \left| \begin{matrix} V(z) \\ I(z) \end{matrix} \right. = 0 \tag{45}$$

Here $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$ (46)

Analogous to the attenuation constant within the free lossy dielectric:(μ, ε_c)

$$\gamma = j\sqrt{\mu\varepsilon_c} = j\sqrt{\mu(\varepsilon - j\frac{\sigma}{\omega}}$$

where $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega}$ $\Leftrightarrow$ attenuation constant (47)

$\beta = \mathrm{Im}\{\gamma\} \mathrm{rad/m}$ $\Leftrightarrow$ phase constant (48)

The solutions of (45) give the phasors of V(z) and I(z):

$V(z) = V_0^{+} e^{-\gamma z} + V_0^{-} e^{\gamma z}$ (49.1)

$V(z) = V^{+}(z) + V^{-}(z)$ (49.2)

$V_0^{+} \stackrel{\Delta}{=} \left|V^{+}(0)\right| e^{j\varphi_0^{+}}$ (49.3)

$V(z) \stackrel{\Delta}{=} \left|V^{+}(0)\right| e^{-\alpha z} e^{j(\varphi_0^{+} - \beta z)} + \left|V^{-}(0)\right| e^{\alpha z} e^{j(\varphi_0^{-} + \beta z)}$ (49.4)

I(z) has the same properties as V(z)

$I(z) = I^{+}(z) - I^{-}(z)$ (50)

The relation between V(z),I(z) waves can be found by substituting (49) and (50) in (43) and (44);

$$Z_0 \stackrel{\Delta}{=} \frac{V^{+}(z)}{I^{+}(z)} = +\frac{V^{-}(z)}{I^{-}(z)} = \frac{R + j\omega L}{\gamma} \Rightarrow \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}\,\Omega \quad (51)$$

IMPORTANT SPECIAL CASES

(1) LOSSLESS LINE $\Leftrightarrow$ (R=0 , G=0)

(a) PROPAGATION CONSTANT:

$\gamma=\alpha+J\beta=J\omega\sqrt{LC}$ $\alpha=0$ $\Rightarrow$ zero attenuation and
$\beta=J\omega\sqrt{LC}$ $\Rightarrow$ β is the lineer function of the ω ; (52)

(b) $U_p=\dfrac{\omega}{\beta}=\dfrac{1}{\sqrt{LC}}=CONST.$ $\Rightarrow$ (all the frequency combinations of a signal packet will have the same u_p speed along the line);

(c) CHARACTERISTIC IMPEDANCE :

$$Z_0=R_0+jX_0=\sqrt{\frac{L}{C}}\Omega, R_0=\sqrt{\frac{L}{C}}\Omega \tag{53}$$

(2) LOW LOSS LINE $\Leftrightarrow$ (R<<ωL , G<<ωC)

(a) PROPAGATION CONSTANT : γ

$$\gamma=\alpha+J\beta=J\omega\sqrt{LC}\left(1+\frac{R}{J\omega L}\right)^{1/2}\left(1+\frac{G}{J\omega C}\right)^{1/2}$$

From the binomial series expansion using for

$$\frac{R}{\omega L} << 1, \frac{G}{\omega C} << 1$$

$\alpha=\dfrac{1}{2}\left(R\sqrt{\dfrac{L}{C}}+G\sqrt{\dfrac{L}{C}}\right)$ or $\alpha=\dfrac{1}{2}\left(\dfrac{R}{Z_0}+GZ_0\right)$ Np/m $\Rightarrow$ (all the frequency combinations of a signal packet will have the same amount of attenuation along the line);
$\beta=\omega\sqrt{LC}$ rad/m $\Rightarrow$ β is the lineer function of the ω ; (54)

(b) $U_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$) $\Rightarrow$ (all the frequency combinations of a signal packet will have the same u_p speed along the line);

(c) $\mathbf{Z_0=R_0+JX_0}=\sqrt{\frac{L}{C}}\left(1+\frac{R}{J\omega L}\right)^{1/2}\left(1+\frac{G}{J\omega C}\right)^{1/2}$

From the binomial series expansion using for $\frac{R}{\omega L} << 1, \frac{G}{\omega C} << 1$;

$$R_0 = \sqrt{\frac{L}{C}}, X_0 = -\sqrt{\frac{L}{C}}\frac{1}{2\omega}\left(\frac{R}{L}-\frac{G}{C}\right) \cong 0 \qquad (55)$$

(3) DISTORTIONLESS LINES $\Leftrightarrow \frac{R}{L} = \frac{G}{C}$

(a) PROPAGATION CONSTANT:

$$\gamma=\alpha+j\beta=(R+j\omega L)^{1/2}\left(J\omega C+\frac{RC}{L}\right)^{1/2} = \sqrt{\frac{C}{L}}(R+j\omega L)$$

$\alpha = R\sqrt{\frac{C}{L}} \Rightarrow$ independent of ω , $\beta = \omega\sqrt{LC} \Rightarrow$lineer function of ω ; (56)

(b) $U_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \Rightarrow$ independent of ω

(c) CHARACTERISTIC IMPEDANCE:

$$z_0=r_0+jx_0=\sqrt{\frac{R+J\omega L}{\frac{RC}{L}+J\omega C}} = \sqrt{\frac{L}{C}}; R_0 = \sqrt{\frac{L}{C}}; X_0 = 0 \qquad (57)$$

HOMEWORK

Z_0=50 Ω ; DISTORTIONLESS LINE ; α=0.01 dB/m ; C=0.1 pF/m are .given

(a) The other line distributed parameters and phase velocity are required ;(Result :R(Ω/m) $\Rightarrow$ 0.057 Ω/m ; L(H/m) $\Rightarrow$ 0.25 μH/m ; G(μS/m) $\Rightarrow$ 2.28 μS/m and U_p= 2x 10^8 m/sn)

(b) At the distances of l_1=1 km , l_2= 5 km ; find out attenuations as the ratio ?

Hint : $\alpha = \mathrm{Re}\{\gamma\} = \mathrm{Re}\left\{\sqrt{(R+J\omega L)(G+J\omega C)}\right\}$ (Np/m

α FORMULA USING THE POWER RELATIONS ON A REFLECTIONLESS LINE

For infinite length line ,or finite line terminated by z_0 , the voltage ,current and power waves can be expressed as;

$$v(z) = v_0 e^{-(\alpha+j\beta)z}$$

$$I(z) = \frac{V_0}{Z_0} e^{-(\alpha+J\beta)z} \tag{58}$$

$$P(z) = \frac{1}{2}\mathrm{Re}\left(V(z)I(z^*)\right) = \frac{|V_0|^2}{2|Z_0|^2} R_0 e^{-2\alpha z} \tag{59}$$

From the law of energy conservation,one can write

$$-\frac{\partial P(z)}{\partial z} = P_L = 2\alpha P(z) \qquad \Rightarrow \qquad \alpha = \frac{P_L(z)}{2P(z)} \text{Np} / \text{m} \tag{60}$$

Calculating $P_l(z)$ by using the lossy equivalent circuit,we have

$$P_L(z) = \frac{1}{2}\left[|I(z)|^2 R + |V(z)|^2 G\right]$$

$$P_L(z) = \frac{V_0^{\ 2}}{2|Z_0|^2}(R + G|Z_0|^2)e^{-2\alpha z} \qquad (61)$$

Substituting (58),(59),(60) ,one obtains;

$$\alpha = \frac{1}{2R_0}\left(R + G|Z_0|^2\right) \text{ np/m} \qquad (62)$$

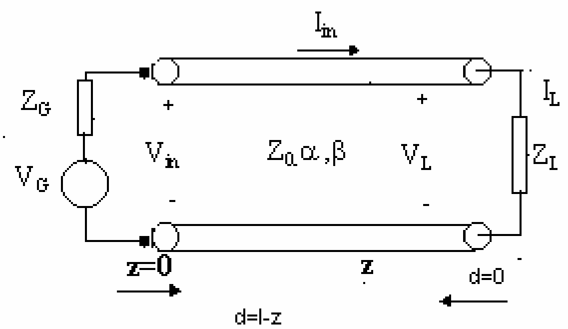
For low loss line , using $Z_0 = R_0 = \sqrt{\frac{L}{C}}$

$$\alpha = \frac{1}{2}\left(\frac{R}{R_0} + GR_0\right) = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) \qquad (63)$$

Distortionless line $Z_0 = R_0 = \sqrt{\frac{L}{C}}$, using $\frac{R}{L} = \frac{G}{C}$

$$\alpha = \frac{1}{2}R\sqrt{\frac{C}{L}}\left(1 + \frac{GL}{RC}\right) \Rightarrow \alpha = \frac{1}{2}R\sqrt{\frac{C}{L}} = \frac{R}{R_0} \qquad (64)$$

GENERAL CASE : A TRANSMISSION LINE TERMINATED BY AN ARBITRARY IMPEDANCE



$$V(z) = V^+(z) + V^-(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \qquad (65)$$

$$V(z) = \left|V_0^+\right| e^{-\alpha z} e^{-j(\beta z - \varphi_0^+)} + \left|V_0^-\right| e^{-\alpha z} e^{j(\beta z + \varphi_0^-)} \qquad (66)$$

$$I(z) = I^+(z) - I^-(z) = \frac{\left|V_0^+\right|}{\left|Z_0\right|} e^{-\alpha z} e^{-j(\beta z - \varphi_0^+ + \varphi_{Z0})} - \frac{\left|V_0^-\right|}{\left|Z_0\right|} e^{\alpha z} e^{j(\beta z + \varphi_0^- - \varphi_{Z0})}$$

(67)

REFLECTION COEFFICIENT FUNCTION

$\Gamma(z) \stackrel{\Delta}{=}$ The reflected component of voltage (current)
The coming component of voltage (current) using this definition , let us find out $\Gamma(z)$ for $(z=\ell)$:

$$\Gamma_L \stackrel{\Delta}{=} \Gamma(\ell) = \frac{V_0^- e^{\gamma\ell}}{V_0^+ e^{-\gamma\ell}} = \frac{V_0^- e^{2\gamma\ell}}{V_0^+} \qquad (68)$$

and $\Gamma(z)$ function can be written as

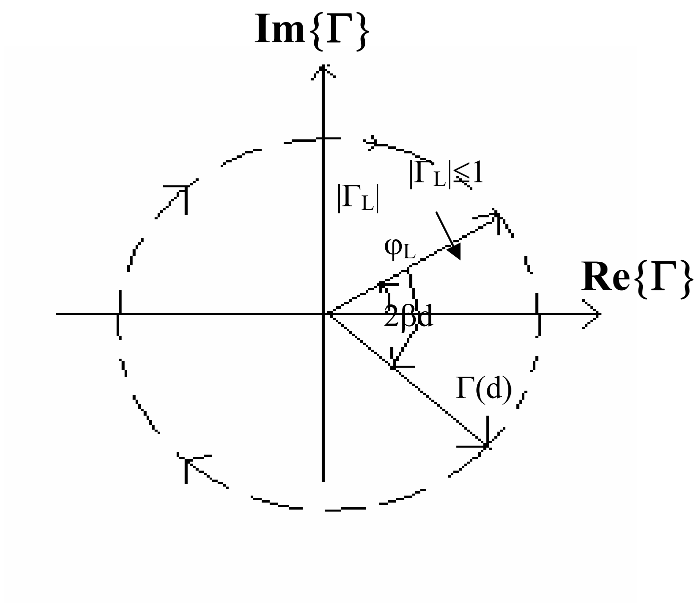
$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{2\gamma z} \qquad (69)$$

so $\Gamma(z)$ can be expressed in terms of Γ_1 using (68) :

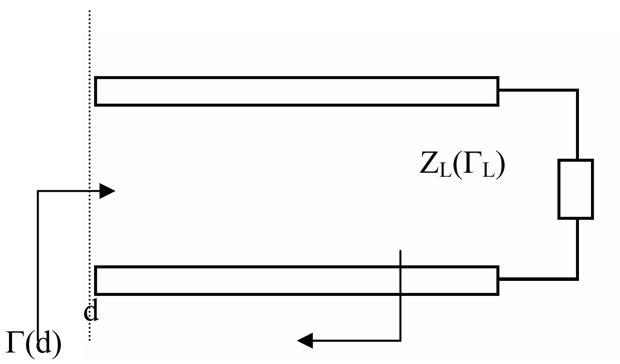
$$\Gamma(z) = \Gamma_L e^{-2\gamma(\ell - z)} \Leftrightarrow \Gamma(d) = \Gamma_L e^{-2\gamma d} \qquad (70\ )$$

$$\Gamma(d) = \left|\Gamma_L\right| e^{-2\alpha d} \left| \varphi_L - 2\beta d \right.$$

If a Γ_1 is given, we can find out $\Gamma(d)$ for lossless line as shown:



(1) Take α=0 (lossless line) $\Gamma(d) = |\Gamma_L| e^{-2\alpha d} | \varphi_L - 2\beta d$ (71)



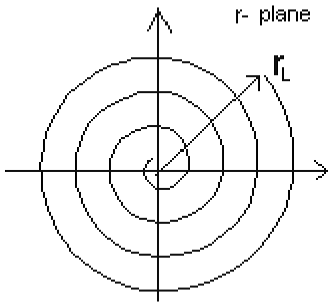
$$\Gamma(d) = \frac{V^-(d)}{V^+(d)} \tag{72}$$

$|\Gamma(d)| = |\Gamma_L| \Rightarrow$CIRCLE (73)

$\varphi_r(d) = \varphi_L - 2\beta d$ (74)

(2) $\alpha \neq 0$, $|\Gamma(d)| = |\Gamma_L| e^{-2\alpha d} \Rightarrow$ (SPIRAL)

$$\varphi_r(d) = \varphi_L - 2\beta d \tag{75}$$



So if you go towards the source from the load ,all the $\Gamma(d)$ take place on the spital starting from the Γ_l ending to the $\Gamma(\ell)$.

Let's find V(z) and I(z) using $\Gamma(z)$:

$$V(z) = V_0^+ e^{-\gamma z}(1+\Gamma(z)) = V_0^+ e^{-\gamma z}(1+\Gamma_L e^{-2\gamma(\ell - z)}) \tag{76}$$

$$I(z) = \frac{V_0^+ e^{-\gamma z}}{Z_0}(1-\Gamma(z)) = \frac{V_0^+ e^{-\gamma z}}{Z_0}(1-\Gamma_L e^{-2\gamma(\ell - z)}) \tag{77}$$

Let us write the boundary condition for $z = \ell$

$$V_L = V(\ell) = V_0^+ e^{-\gamma \ell}(1+\Gamma_L) \tag{78}$$

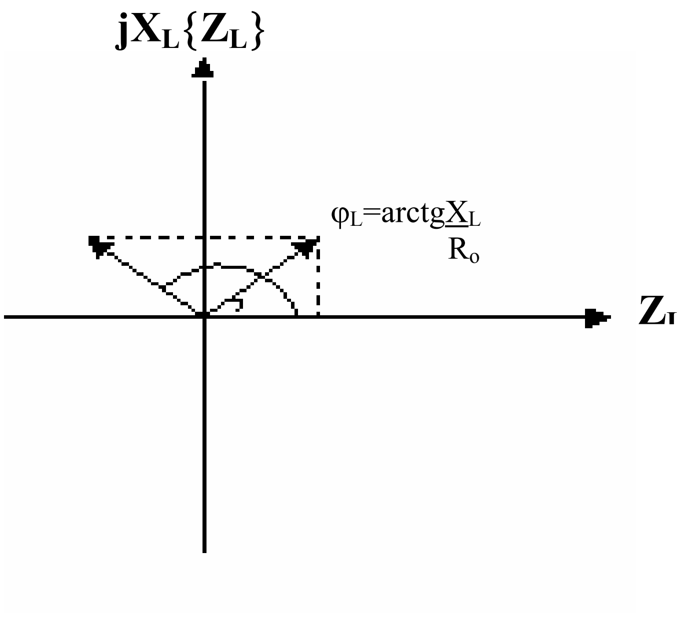
$$I_L = I(\ell) = \frac{V_0^+ e^{-\gamma \ell}}{Z_0}(1-\Gamma_L) \tag{79}$$

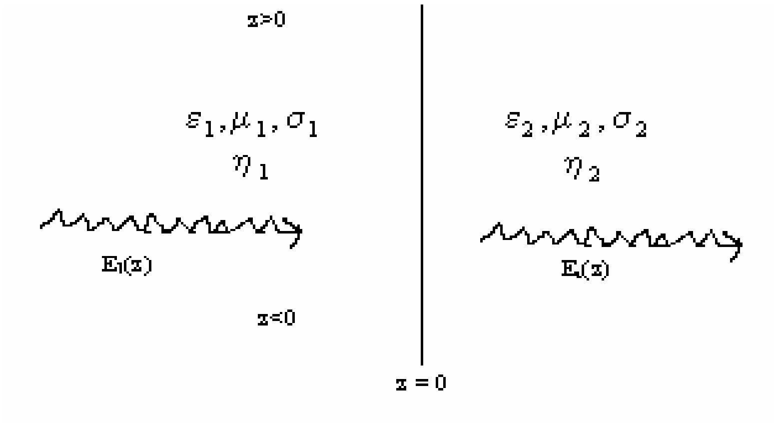
$$Z_L \stackrel{\Delta}{=} \frac{V_L}{I_L} \Rightarrow V_L = Z_L . I_L \tag{80}$$

Substitute (78) and (79) at (80);

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (81)$$

- For $Z_L = Z_0$,$\Gamma_L = 0 \Leftrightarrow \Gamma(Z) \Leftrightarrow V^-(Z) = I^-(Z) = 0$
- $|\Gamma_L| \leq 1$
- $Z_L = 0 \Rightarrow \Gamma_L = -1 \Rightarrow V^-(\ell) = -V^+(\ell)$
- $Z_L \to \infty$,$\Gamma_L \to 1$ $\quad V^-(\ell) = -V^+(\ell)$ $\quad (R_L > R_0)$
- IF $Z_L = R_L \Rightarrow \Gamma_L = \dfrac{R_L - R_0}{R_L + R_0}$
- $Z_L = JX_L \Rightarrow \Gamma_L = \dfrac{JX_L - R_0}{JX_L + R_0} = \dfrac{\left|\mathrm{arctg}\dfrac{X_L}{-R_0}\right.}{\mathrm{arctg}\dfrac{X_L}{-R_0}}$





$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}};\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}};\varepsilon_{c1} = \varepsilon_1\left(1 - j\frac{\sigma_1}{\omega}\right);\varepsilon_{c2} = \varepsilon_2\left(1 - j\frac{\sigma_2}{\omega}\right)$$

(82)

$$\Gamma \stackrel{\Delta}{=} \frac{E_r(0)}{E_i(0)} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Standing waves pattern⇔ |v(z)| - z

α=0⇔ lossless line using $v(z)=v_0^{+}e^{-j\beta z}(1+\Gamma(z))$

$$\left|\frac{V(z)}{V_0^{+}}\right| = \left|1+\Gamma(z)\right| = \left|1+\Gamma_L e^{-J2\beta d}\right| = \left|\left(\left|\Gamma_L\right|\cos(-2\beta d+\varphi_L\right)+J\left(\left|\Gamma_L\right|\sin(-2\beta d+\varphi_L\right)\right|$$

(83)

Finding the maximums and minimums of $\left|\frac{V(z)}{V_0^{+}}\right|$;

$$\left|\frac{V(d)}{V_0^{+}}\right| = \sqrt{(1+\left|\Gamma_L\right|\cos(-2\beta d+\varphi_L))^2 + \left|\Gamma_L\right|^2\sin^2(-2\beta d+\varphi_L)} \tag{84}$$

$$\left|\frac{V(d)}{V_0^+}\right| = \sqrt{(1 + 2|\Gamma_L|\cos(-2\beta d + \varphi_L) + |\Gamma_L|^2} \qquad (85)$$

$-2\beta d_{max}+\varphi_l= n2\Pi \qquad n=0,\pm1, \pm2,.............$

$$\left|\frac{V(d_{max})}{V_0^+}\right|\Big|\max = 1 + |\Gamma_L|$$

$$d_{max} = n\frac{\lambda}{2} + \frac{\varphi_L}{4\Pi}\lambda \qquad (86)$$

$$|V(z)|_{max} = |V_0^+|(1 + |\Gamma_L|) \qquad (87)$$

$-2\beta d_{min}+\varphi_l= (2n+1)\Pi \qquad n=0,\pm1, \pm2,.............$

$$d_{min} = -(2n+1)\frac{\lambda}{2} + \frac{\varphi_L}{4\Pi}\lambda \qquad (88)$$

Between two maxima or minima :$\lambda/2$;
Two maxima and minima : $\lambda/4$

$$|V(d)|_{min} = |V_0^+|(1 - |\Gamma_L|) \qquad (89)$$

$$|I(d_{max})| = \left|\frac{V_0^+}{Z_0}\right|(1 - |\Gamma_L|) \qquad (90)$$

$$\text{VSWR} \stackrel{\Delta}{=} \frac{|V|_{max}}{|V|_{min}} = \frac{1 + \Gamma_L}{1 - \Gamma_L} \qquad (91)$$

Here $$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1} \quad ; z_L = \frac{Z_L}{Z_0} \tag{92}$$

GRAPHICAL APPROACH

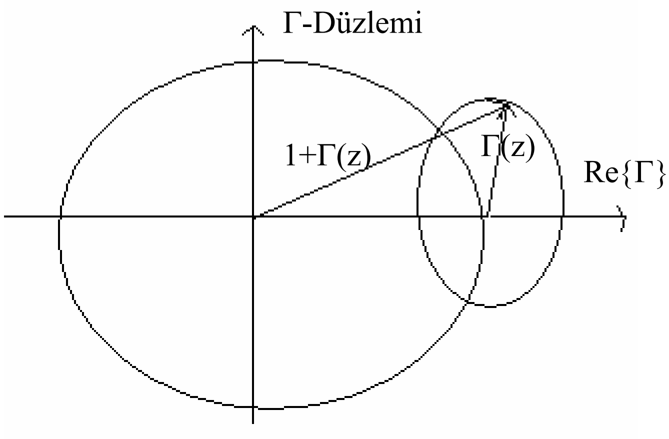
α=0 $\Leftrightarrow$ lossless line

$$\left|\frac{V(z)}{V_0^+}\right| = \left|1 + \Gamma(z)\right| = \left|1 + \Gamma_L e^{-j2\beta(\ell - z)}\right|$$

$$\varphi_\Gamma(z) \overset{\Delta}{=} 2\beta(\ell - z) + \varphi_L \tag{93}$$

$$\varphi_{\Gamma\max}(z) = n2\pi \text{....} n = 0, \mp 1, \mp 2.. \tag{94}$$

$$\left|\frac{V(z)}{V_0^+}\right| = \left|1 + \Gamma(z)\right| \tag{95}$$



$$\varphi_{\Gamma\max}(z) = (2n+1)\pi, n = 0, \pm 1, \pm 2.. \tag{96}$$

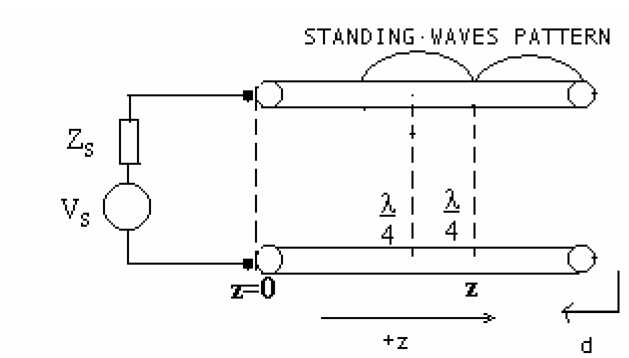
$$\left|\frac{V(z)}{V_0^{\,+}}\right|_{\min} = \left|1 - \Gamma(z)\right| \tag{97}$$

HOMEWORK:
Find the maximum and minimum positions and values of |I(z)| - z

$$\varphi_\Gamma(z) \stackrel{\Delta}{=} \varphi_L + -2\beta d \qquad \Gamma_L \stackrel{\Delta}{=} \frac{Z_L - Z_0}{Z_L + Z_0} = \left|\Gamma_L\right| \angle \varphi_L \tag{98}$$

For Special Terminations standing waves pattern and VSWR:

(1) Open-circuited termination: $Z_L \to \infty \Rightarrow \Gamma_L = 1 = 1\angle 0^0$

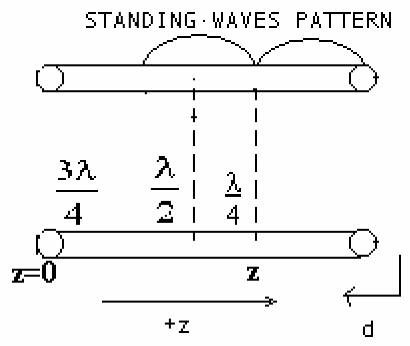


$$\varphi_\Gamma(d)\Big|_{\max} = +2\beta d = n2\pi, d_{\max} = \frac{n\lambda}{2}, d_{\max 1} = 0, d_{\max 2} = \frac{\lambda}{2}, d_{\max 3} = \lambda \tag{99}$$

$$\varphi_\Gamma(d)\Big|_{\min} = +2\beta d_{\min} = (2n+1)\pi, d_{\min} = \frac{(2n+1)\lambda}{2}, d_{\min 1} = \frac{\lambda}{4}, d_{\min 2} = \frac{3\lambda}{4} \tag{100}$$

$$\text{VSWR} = \frac{1+\Gamma_L}{1-\Gamma_L}\Big|\Gamma_L \to 1$$ **;FOR OPEN CIRCUIT TERMINATED**

$$\left|\frac{V(z)}{V_0^{+}}\right| = \sqrt{2+2\cos 2\beta d}$$

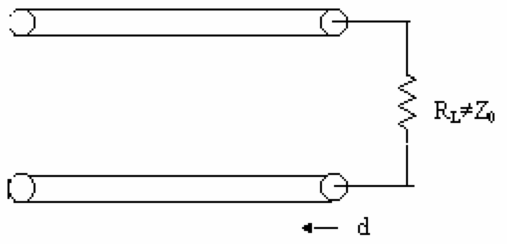


HOMEWORKS:

Find the standing waves pattern and VSWR for the above terminations

1. $|v(z)|=|v^{+}|$ $z_l=z_0$
2. $Z_l \to 0$(short cırcuıt)
3. $Z_l=R_l+jX_l$
4. $Z_l \to 0 \;\to \Gamma_l=0$;VSWR=1;$|V(z)|_{max}=|v(z)_{min}$
5. $Z_l=R_l$
6. $Z_l=jX_l$

$$\text{VSWR}=\frac{|V(z)|_{max}}{|V(z)|_{min}} = \frac{1+\Gamma_L}{1-\Gamma_L} \Rightarrow \Gamma_L = \frac{\text{VSWR}-1}{\text{VSWR}+1}$$



(5) $Z_l=R_l$
Terminating line with pure rezistance

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

1) $R_l > Z_0 \Rightarrow \Gamma_l = |\Gamma_l| \llcorner 0^0$

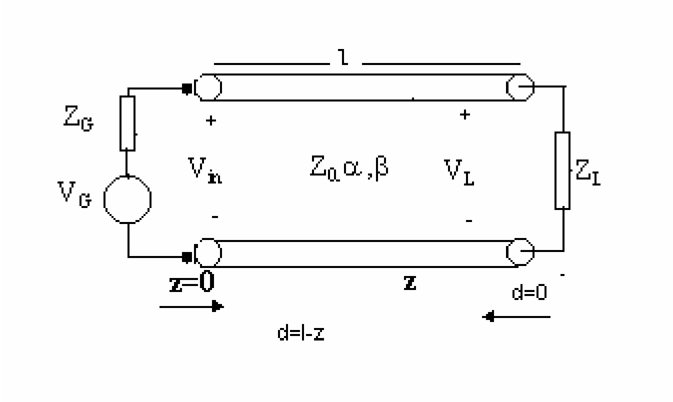
2) $R_l < Z_0 \Rightarrow \Gamma_l = |\Gamma_l| \llcorner \Pi$

$$\varphi_\Gamma(d) \overset{\Delta}{=} \varphi_L + -2\beta d$$

1) $\varphi_\Gamma(d) \overset{\Delta}{=} -2\beta d$ $\quad |V_{max}|=?,\ |V_{min}|=?,\ VSWR=?$

2) $R_l < Z_0$, $\varphi_\Gamma(d) \overset{\Delta}{=} \Pi - 2\beta d \quad \Pi \overset{\Delta}{=} \Pi - 2\beta d_{min}, d_{min} = 0$
$V_{min}|=?$,VSWR=?

POWER FLOW ALONG THE TERMINATED LINE



The net power P(z) at a z-position of the line

$$P(z)=\mathrm{Re}\{V(z)I^*(z)\} \tag{101}$$

where $V(z)=V^+(z)(1+\Gamma(z))$ and $I(z) = \dfrac{V^+(z)}{Z_0}(1-\Gamma(z))$

$$V(z)=V^+(z)e^{-\gamma z} \tag{102}$$

$$P(z) = \mathrm{Re}\left\{V^+(z)(1+\Gamma(z))\frac{\left(V^+(z)\right)^*}{Z_0^{\ *}}(1-\Gamma^*(z))\right\} \quad \text{(for low loss line)}$$

$$P(z) = \mathrm{Re}\left\{ \frac{\left|V^+(z)\right|^2}{Z_0}(1 - \left|\Gamma(z)\right|^2) - \Gamma^*(z) + \Gamma(z) - \Gamma + J\Gamma_i + \Gamma + J\Gamma_i \right\}$$

$$P(z) = \mathrm{Re}\left\{ \frac{\left|V^+(z)\right|^2}{Z_0}(1 - \left|\Gamma(z)\right|^2) \right\} \qquad (103.1)$$

$$P^+(z) \stackrel{\Delta}{=} \mathrm{Re}\left\{ V^+(z)\left(I^+(z)\right)^* \right\} = \frac{\left|V^+(z)\right|^2}{Z_0} \qquad (103.2)$$

$$P^-(z) \stackrel{\Delta}{=} \frac{\left|V^-(z)\right|^2}{Z_0} = \left|\Gamma(z)\right|^2 P^+(z) \qquad (103.3)$$

$$P(z) = P^+(z) + P^-(z)$$

$$P(z) = P^+(z)(1 - \left|\Gamma(z)\right|^2)$$

The net power at the input of the line is

$$P_{in} = P^+{}_{in}\left(1 - \left|\Gamma_{in}\right|^2\right) \qquad (104)$$

where

$$P_{in}{}^+ = \frac{\left|V^+(0)\right|^2}{Z_0} = \frac{\left|V_0{}^+\right|^2}{Z_0} \qquad (105)$$

is the power of the wave that comes to the input of the line. The net power that goes into the load is

$$P_L = P_L^{+}\left(1 - \left|\Gamma_L\right|^2\right) \tag{106}$$

where

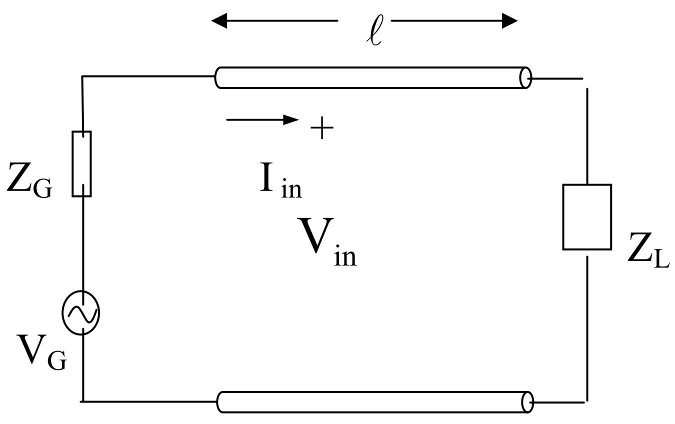
$$P_L^{+} = \frac{\left|V_L^{+}\right|^2}{Z_0} = \frac{\left|V_0^{+} e^{-\gamma\ell}\right|^2}{Z_0} = \frac{\left|V_0^{+}\right|^2 e^{-2\alpha\ell}}{Z_0} \tag{107}$$

is the power of the wave that comes to the load and may also be written as

$$P_L^{+} = P_{in}^{+} e^{-2\alpha\ell} \tag{108}$$

We can generalize Eq.(108)

$$P^{+}(z) = P_{in}^{+} e^{-2\alpha z} \tag{109}$$



From Kirschhoff's voltage law

$$\mathrm{V_G} = \mathrm{V_{in}} + \mathrm{I_{in}} Z_G \tag{110}$$

$$\mathrm{V_G} = \mathrm{V_0^+}(1 + \Gamma_{\mathrm{in}}) + \frac{\mathrm{V_0^+}}{\mathrm{Z_0}} Z_G (1 - \Gamma_{\mathrm{in}}) \tag{111}$$

$$\Gamma_{\mathrm{in}} = \Gamma_{\mathrm{L}} \mathrm{e}^{-2\gamma\ell} \tag{112}$$

$\mathrm{V_0^+}$ is the sum of all the voltage wave components traveling in the +z direction at the z=0 location

$$\mathrm{V_0^+} = \mathrm{V^+}(0) \tag{113}$$

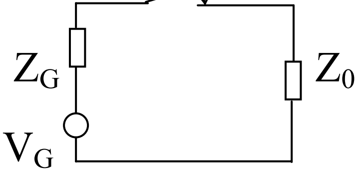
For the reflected wave we consider the reflection coefficient at the source end

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} \tag{114}$$

Solving for Z_G, Eq. (114) becomes

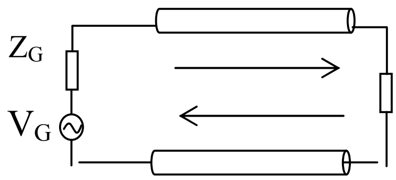
$$Z_G = Z_0\left(\frac{1+\Gamma_G}{1-\Gamma_G}\right) \tag{115}$$

At the time $t = 0^+$



$$V_0^+ = \frac{V_G Z_0}{Z_G + Z_0} \tag{116}$$

Second V_0^+ wave



$$V_0^+ = \Gamma_L \Gamma_G e^{-2\gamma\ell} \frac{V_G Z_0}{Z_G + Z_0} \tag{117}$$

Third V_0^+ wave

$$V_0^+ = (\Gamma_L \Gamma_G e^{-2\gamma\ell})^2 \frac{V_G Z_0}{Z_G + Z_0} \tag{118}$$

If we go on like that, we have

$$V_0^+ \stackrel{\Delta}{=} V^+(0) = \frac{V_G Z_0}{Z_0 + Z_G} \left[1 + \Gamma_G \Gamma_L e^{-2\gamma\ell} + (\Gamma_G \Gamma_L e^{-2\gamma\ell})^2 + \ldots\ldots\ldots\ldots\right] \tag{119}$$

For $\left|\Gamma_G \Gamma_L e^{-\gamma\ell}\right| \langle 1$ the series in Eq. (119) converges.

$$V^+(0) = \frac{V_G Z_0}{Z_0 + Z_G} \sum_{j=0}^{\infty} (\Gamma_G \Gamma_L e^{-2\gamma\ell})^j = \frac{V_G Z_0}{Z_0 + Z_G} \frac{1}{1 - \Gamma_L \Gamma_G e^{-2\gamma\ell}} \tag{120}$$

where

$$Z_0 + Z_G = Z_0 + Z_0 \frac{1+\Gamma_G}{1-\Gamma_G} = Z_0 \frac{2}{1-\Gamma_G} \tag{121}$$

So we obtain

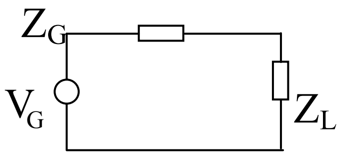
$$V^+(0) = \frac{(1-\Gamma_G)V_G}{2(1-\Gamma_L \Gamma_G e^{-\gamma\ell})} \tag{122}$$

$$P^+(z) = \frac{|V_G|^2}{4Z_0} \left| \frac{1-\Gamma_G}{1-\Gamma_G \Gamma_L e^{-\gamma\ell}} \right|^2 e^{-2\alpha z} \tag{123}$$

$P^+(z)$ is the total power of waves traveling in the $+z$ direction. P_A is the maximum power of load and defined as

$$P_A \stackrel{\Delta}{=} \frac{|V_G|^2}{4R_G} \tag{124}$$

V_G is the rms value and $R_G = \Re e\{Z_G\}$



$$Z_L = Z_G^*$$

$$P_{in}^+ = P_A \frac{1-|\Gamma_G|^2}{|1-\Gamma_G \Gamma_{in}|^2} \tag{125}$$

For a lossy line;

$$P_m = P_{in}^{+}\left(1-\left|\Gamma_{in}\right|^2\right) = P_A \frac{\left(1-\left|\Gamma_G\right|^2\right)\left(1-\left|\Gamma_{in}\right|^2\right)}{\left(1-\Gamma_G\Gamma_{in}\right)^2} \qquad (126)$$

$$P_L = P_{in}^{+} e^{-2\alpha z}\left(1-\left|\Gamma_L\right|^2\right) = P_A \frac{\left(1-\left|\Gamma_G\right|^2\right)\left(1-\left|\Gamma_L\right|^2\right)}{\left|1-\Gamma_G\Gamma_L e^{-2\gamma z}\right|^2}$$

The net power at the z location,

$$P(z) = P^{+}(z)\left(1-\left|\Gamma(z)\right|^2\right)$$
$$P_{in} - P_L \Rightarrow \text{gives lossy factor.}$$

For a lossless line;

$$\alpha = 0$$
$$P_{in} = P_L = P_A \frac{\left(1-\left|\Gamma_G\right|^2\right)\left(1-\left|\Gamma_L\right|^2\right)}{\left|1-\Gamma_G\Gamma_L e^{-j2\beta 2}\right|^2} \qquad (127)$$

SPECIAL CASES

Line is driven by a matched source

$$Z_G = Z_0$$

so

$$\Gamma_G = 0$$

Eq. (126) becomes

$$P_m^+ = P_A = \frac{|V_G|^2}{4Z_0}$$

$$P_{in} = P_A\left(1 - |\Gamma_{in}|^2\right) \tag{128}$$

$$P_L = P_A e^{-2\alpha\alpha}\left(1 - |\Gamma_L|^2\right)$$

If $\Gamma_G = 0,\ \Gamma_L = 0\ \text{and}\ \alpha = 0$

$$P_L = P_{in} = P_A \tag{129}$$

REFLECTION LOSS

$L_R = 10 \cdot \log\left(\frac{P^+}{P^-}\right)$ dB is equation of reflection loss.

$P^- = |\Gamma|^2 \cdot P^+$ thus ;

$L_R = 10 \cdot \log \frac{1}{|\Gamma|^2}$ dB is obtained.

$|\Gamma| = |\Gamma_L| \cdot e^{-2\alpha d}$ is unity of $\alpha \cdot d$ is Neper.

In this case between line of input and load reflection loss is derived that

$$L_R = 10 \cdot \mathbf{log}_{10}\left(\frac{1}{\left|\Gamma_L\right|^2 \cdot e^{-4\alpha d}}\right) = \underbrace{10 \cdot \mathbf{log}_{10}\frac{1}{\left|\Gamma_L\right|^2}}_{\mathbf{I}} + \underbrace{10 \cdot \mathbf{log}_{10}\left(e^{4\alpha d}\right)}_{\mathbf{II}}$$

I : L_R load loss due to the load reflection

II : loss term due to the attenuation of line

$$L_{Rin} = L_{Rload} + 2 \cdot 8{,}686 \cdot \alpha \cdot \ell$$

EXAMPLE : A 500 MHz generator with $V_G = 20\ V_{rms}$ and internal resistance $Z_G = 100\ \Omega$ is connected to a $Z_0 = 100\ \Omega$ transmission line that is $\ell = 4$ m long and terminated in a $Z_L = 150\ \Omega$ load. Find P_{in}

a) For α=0 dB/m and delivered power to load P_L

b) Repeat a) for α=0,5 dB/m

SOLUTION :

Source :

$$P_A = \frac{\left|V_G\right|^2}{4R_G} = \frac{20^2}{4 \times 10^2} = 1\,W$$

P_A= 1 W f = 500 MHz since $Z_G = Z_0 = 100\ \Omega$ so $\Gamma_G = 0$ source is matched to the transmission line.

Load : $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{150 - 100}{250} = 0{,}2$

$$P_{in} = P_{in}^{+} \cdot \left(1 - \left|\Gamma_{in}\right|^{2}\right) \text{ where } ;$$

$$P_{in}^{+} = P_{A} \text{ and } \Gamma_{in} = \Gamma_{L} e^{-2\alpha\ell} = \Gamma_{L} \text{ so that}$$

$$P_{in} = 1 \cdot \left(1 - 0{,}2^{2}\right) = 0{,}96 \text{ W is obtained.}$$

(a) For the lossless line α=0 is given before , so that

P_L =P_{in}=1 W

(b) For the lossy line α=0,5 dB/m

$$\alpha = \frac{0{,}5}{8{,}69} = 0{,}23 \text{ Np/m} \text{ and } 2 \cdot \alpha \cdot \ell = 0{,}46 \text{ Np}$$

$$P_{in} = P_{in}^{+} \cdot \left(1 - \left|\Gamma_{in}\right|^{2}\right) \text{ where};$$

$$P_{in}^{+} = P_{A} \text{ and} \left|\Gamma_{in}\right| = \left|\Gamma_{L}\right| e^{-2 \cdot \alpha \cdot \ell} = 0{,}2\, e^{-0{,}46} = 0{,}126$$

$$P_{in} = 1 \cdot \left(1 - 0{,}2^{2}\right) = 0{,}984 \text{ W is found.}$$

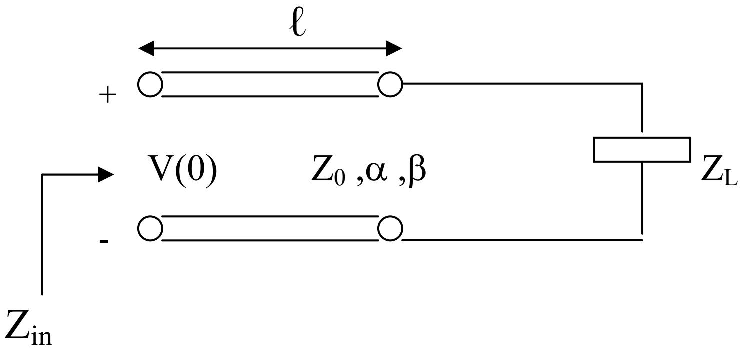
$$P_{L} = P_{L}^{+} e^{-2\alpha\ell} \left(1 - \left|\Gamma_{L}\right|^{2}\right) = 1.e^{-0{,}46}\left(1 - 0{,}2^{2}\right) = 0{,}605 \text{ W}$$

$$P_{loss} = P_{in} - P_{L} = 0{,}984 - 0{,}605 = 0{,}379 \text{ W}$$

Reflection loss $L_{R} = L_{RLoad} + 8{,}696 \cdot \alpha\ell$

UTILIZATION OF A TRANSMISSION LINE AS A CIRCUIT COMPONENT

INPUT IMPEDANCE OF A TRANSMISSION LINE (Z_{in})



$$\boxed{\theta = \beta \ell}$$

Definition :

$$Z_{in} \overset{\Delta}{=} \frac{V_{in}}{I_{in}} = \frac{V(0)}{I(0)}$$

$$Z_{in} = \frac{V^{+}(1+\Gamma_{in})}{I^{+}(1-\Gamma_{in})} = Z_o(\frac{1 + \Gamma_{in}}{1 - \Gamma_{in}})$$

$$Z_{in} = Z_0 (\frac{1 + \Gamma_{in}}{1 - \Gamma_{in}})$$

if $\alpha = 0$, $Z_0 = R_0$ and substituting $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

we can write ;

$$Z_{in} = Z_0 \left(\frac{1+\Gamma_L e^{-j2\beta\ell}}{1-\Gamma_L e^{-j2\beta\ell}} \right)$$

$$= Z_0 \frac{1 + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2\beta\ell}}{1 - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2\beta\ell}}$$

$$Z_{in}(\beta\ell, Z_L) = Z_0 \left(\frac{Z_L + jZ_0 \mathrm{tg}(\beta\ell)}{Z_0 + jZ_L \mathrm{tg}(\beta\ell)} \right) \qquad (130)$$

- $Z_{in}(\beta\ell, Z_L)$ has the period of either $\theta = \beta\ell + n\pi$ or $\ell = \ell + n\frac{\lambda}{2}$ .In the other words Z_{in} is repaeted by $n\frac{\lambda}{2}$ intervals.

Line impedances at the maxima and minima:

- $$Z_{inMAX} = Z_0 \left(\frac{1+|\Gamma_{in}|}{1-|\Gamma_{in}|} \right) = Z_0 \, \mathrm{VSWR}$$

(resistive)

- $$Z_{inMiN} = Z_0 \left(\frac{1+|\Gamma_{in}|}{1-|\Gamma_{in}|} \right) = \frac{Z_0}{VSWR}$$

IMPORTANT TERMINATIONS

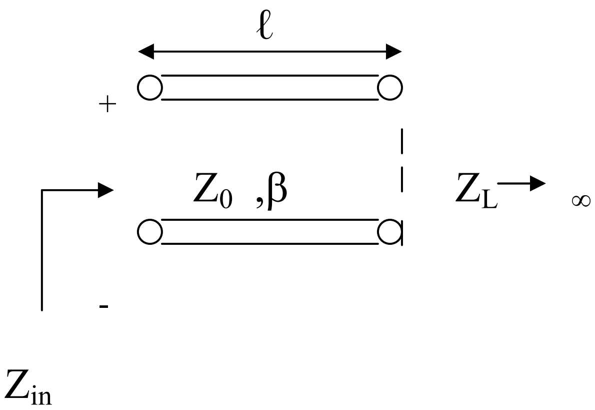
1- OPEN CIRCUIT TERMINATION $(\Gamma_L = 1,\ Z_L \longrightarrow \infty)$

Substituting $Z_L \longrightarrow \infty$ in equation (1), then we have

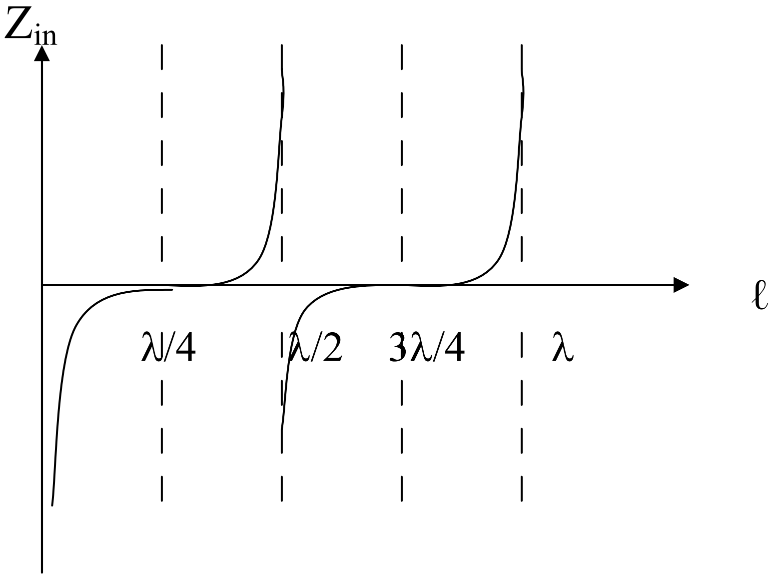
$$Z_{in}(\beta\ell) = Z_0 \left(\frac{Z_L + jZ_0 \operatorname{tg}(\beta\ell)}{Z_0 + jZ_L \operatorname{tg}(\beta\ell)} \right)$$

$$= Z_0 \left(\frac{1 + j\dfrac{Z_0 \operatorname{tg}(\beta\ell)}{Z_L}}{\dfrac{Z_0}{Z_L} + j\operatorname{tg}(\beta\ell)} \right)$$

$$\boxed{Z_{in} = jX_{in} = -jZ_0 \cot(\beta\ell)} \qquad (131)$$



$$Z_{in} = -jZ_0 \cot(\beta \ell)$$



- In case of open circuit termination Z_{in} is purely reactive,
- Z_{in} may be capacitive or inductive depending on $\theta \overset{\Delta}{=} \beta\ell$
- $0 < \ell < \dfrac{\lambda}{4} \longrightarrow Z_{in}$ capacitive
- $\ell = \dfrac{\lambda}{4} + n\dfrac{\lambda}{2} \longrightarrow Z_{in}$ SERIES resonance
- $\dfrac{\lambda}{4} < \ell < \dfrac{\lambda}{2} \longrightarrow Z_{in}$ inductive
- $\ell = \dfrac{\lambda}{2} + n\dfrac{\lambda}{2} \longrightarrow Z_{in}$ PARALEL resonance

- $Z_{in} = -jZ_0 \cot(\beta\ell)$ varies with respect to the frequency because of β.

- If $\beta\ell$ <<1 then $tg(\beta\ell) \cong \beta\ell$ and Z_{in} becomes

$$Z_{in} = jX_{in} = -j\frac{Z_0}{\beta\ell} = \frac{-j\sqrt{\frac{L}{C}}}{\omega\sqrt{LC}\,\ell} = -j\frac{1}{\omega CL}$$

$$\boxed{Z_{in} = -j\frac{1}{\omega CL}}$$

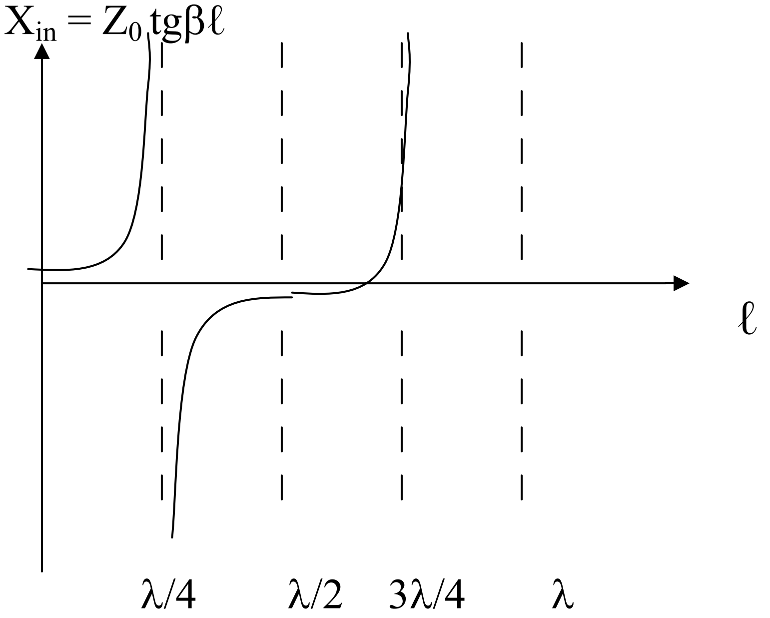
- At microwave frequencies it is not possible to obtain $Z_L \longrightarrow \infty$ because of the coupling to the nearby objects and radiation.

2- SHORT CIRCUIT TERMINATION ($\Gamma_L = -1$, $Z_L = 0$)

$$Z_{in(\beta\ell)} = Z_0\left(\frac{Z_L + jZ_0\, tg(\beta\ell)}{Z_0 + jZ_L\, tg(\beta\ell)}\right)$$

$$= +jZ_0\, tg(\beta\ell)$$

$$Z_{in} = jX_{in} = jZ_0\, tg(\beta\ell)$$



- $\ell = 0 \longrightarrow Z_{in}$ short circuit
- $0 < \ell < \frac{\lambda}{4} \longrightarrow Z_{in}$ inductive
- $\ell = \frac{\lambda}{4} \longrightarrow Z_{in}$ PARALEL resonance
- $\frac{\lambda}{4} < \ell < \frac{\lambda}{2} \longrightarrow Z_{in}$ capacitive
- $\ell = \frac{\lambda}{2} \longrightarrow Z_{in}$ SERIES resonance

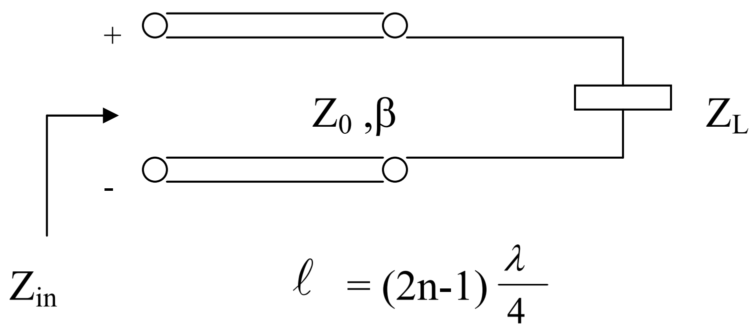
3-QUARTER WAVE LINE TRANSFORMATOR

$\Leftrightarrow \ell = (2n-1)\dfrac{\lambda}{4}$,n = 1, 2,.....=> $\beta\ell = (2n-1)\,.\,\dfrac{\pi}{2}$

Substituting $tg\ (\beta\ell) \longrightarrow \infty$ in equation (130), Z_{in} becomes

$$Z_{in} = \frac{Z_0^{\ 2}}{Z_L} \qquad (132)$$

- Quarter wave line transformator can be used as an impedance invertor.



- $Z_L \longrightarrow \infty$, $Z_{in} = 0$
- $Z_L \longrightarrow 0$, $Z_{in} \longrightarrow \infty$
- $Z_L = J\omega\ L$, $Z_{in} = \dfrac{Z_0^{\ 2}}{j\omega L} = -\ j\,\dfrac{Z_0^{\ 2}}{\omega\ L}$
- $Z_L = \dfrac{1}{j\omega C}$, $Z_{in} = -\ j\omega C\ Z_0^{\ 2}$

Impedance matching in case of $Z_L = R_L$ **:**

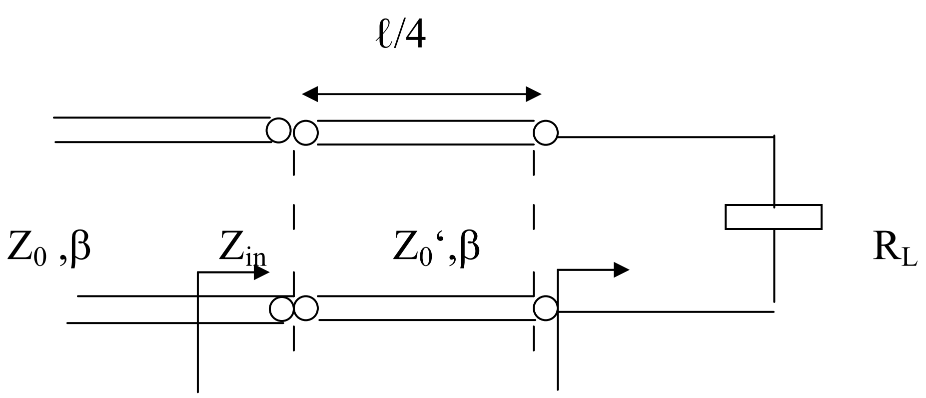
$$Z_{in} = \frac{Z_0^{\;2}}{Z_L} = Z_0 \qquad \Rightarrow \qquad Z_0 = \sqrt{Z_0 . R_L}$$

4-HALF WAVE LINE TRANSFORMATOR

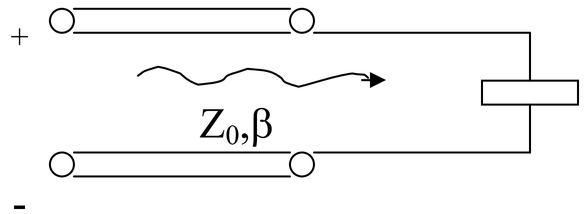
$$\Leftrightarrow \quad \ell = n\frac{\lambda}{2} \quad n = 0,\ 1,\ 2,..... \qquad \beta\ell = n\,\frac{2\pi}{\lambda}\,\frac{\lambda}{2} = n\,\pi$$

$\boxed{Z_{in} = Z_L}$ (133) repeated by $n\,\pi$ intervals.

EXAMPLE : QUARTER WAVE TRANSFORMATOR IN IMPEDANCE MATCHING



The equivalent of the circuit is:



$$Z_{in} = \frac{Z_0'^2}{R_L}$$

$$\Gamma_L = \frac{\dfrac{Z_0'^2}{R_L} - Z_0}{\dfrac{Z_0'^2}{R_L} + Z_0}$$

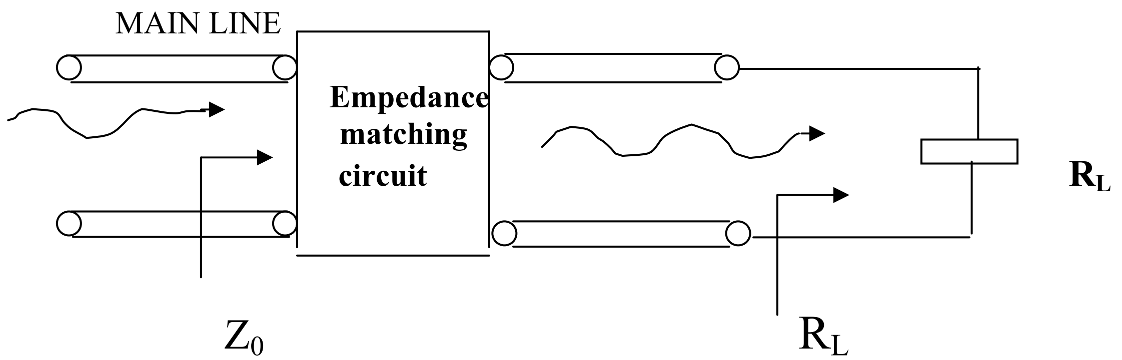
If $\Gamma_L = 0$, then all the power of the incident wave is transferred to the load, so we can write

$$\Gamma_L = 0 \;\; ; \;\; |\Gamma_L| = 0 \qquad P^- = |\Gamma_L|^2 P^+ = 0 \quad \text{(no reflected power)}$$

To obtain $\Gamma_L = 0$ we must have $\dfrac{Z_0'^2}{R_L} = Z_0 \Rightarrow Z_0' = \sqrt{Z_0.R_L}$

where R_L and Z_0 are given.

General Block Diagram of The Impedance Matching

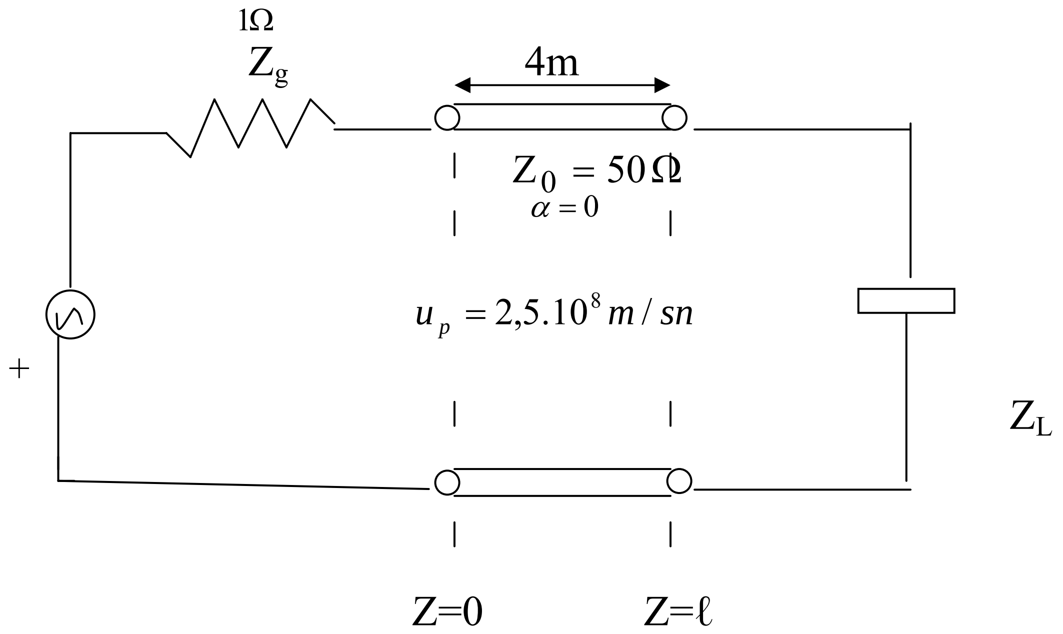


WORKED EXAMPLES :

EXAMPLE 1: For an matched load in any position ; find :

(a) V(z,t) , I(z,t) (b) $V_L(t)$, $I_L(t)$ (c) $P^+(z)$, $P^-(z)$, P (z)

(d) Γ_L , Standing Wave Pattern ,VSWR



Solution: As we have a matched load we must have a Z_L equal to Zo;

$$Z_L = Z_o \quad \Rightarrow \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0,$$

$$VSWR = \frac{1+\Gamma_L}{1-\Gamma_L} = 1$$

$$V(z) = V^+(z) + V^-(z) = V^+(z) \quad (1+ \Gamma(z)) = V^+(z)$$

$$V^{+}(z) = \frac{V_G}{Z_0 + Z_G} Z_0 = 0,3 \, e^{-j\beta z} \frac{50}{51} = 0,294 \, e^{-j}$$

$$\beta = \frac{w}{u_p} \frac{2\pi \times 10^8}{2,5 \times 10^8} = 0,8\pi$$

$$V^{+}(z) = 0,294 \, e^{-j(0,8\pi)z}$$

$$I(z) = \frac{V^{+}(z)}{Z_0} (1 - \Gamma(z))$$

as we have $\Gamma(z) = 0$ then $I^{+}(z) = \dfrac{V^{+}(z)}{Z_0}$

$$I(z) = I^{+}(z) = \frac{0,294}{50} e^{-j(0,8\pi)z} = 5,88.\,10^{-3} \, e^{-j(0,8\pi)z}$$

$$V_L(t) = 0,294 \cos(2\pi \times 10^8 - 3,2\pi)$$

$$I_L(z) = \frac{V_L(z)}{Zo} = 5,88 \times 10^{-3} \, e^{-j3,2\pi}$$

$$V_L(t) = 5,88 \times 10^{-3} \cos(2\pi \times 10^8 - 3,2\pi)$$

c) $\left| \dfrac{V(z)}{V_0^{+}} \right| = \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d)}$

as we have $\Gamma_L = 0$ then

$$\left|\frac{V(z)}{V_0^+}\right| = 1 \qquad\qquad |V(z)| = |V_0^+|$$

* If the source is matched to the line $\Rightarrow Z_G = Z_0$; $\Gamma_G = 0$
the wave carries all the available power of the source;

EXAMPLE 2: Vg=100V , Zg=50Ω , f =10^8 , R_0=50Ω

Z_L=25 +j 25 ve l = 3,6 m are given. Find;

a) V(z) b) $V_İ$ c)V_L d)VSWR e) P_L=?

<u>Solution:</u> As we have $Z_o = Z_G$ so the source is matched to the line in this case we have $\Gamma_G = 0$.

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma(z)) \qquad \Gamma(z) = \Gamma_L e^{-j2\beta d}$$

$$V(z) = \frac{V_G}{Z_0 + Z_G} Z_G . e^{-j\beta z} (1 + \Gamma_L e^{-j2\beta d})$$

where

$$V_0^+ = \frac{V_G}{Z_0 + Z_G} Z_G$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25 + j25 - 50}{25 + j25 + 50} = \frac{-25 + j25}{75 + j25}$$

$$= \frac{-35 e^{-j135}}{79 e^{j18,43}}$$

$$\Gamma_L = 0,44 e^{j116,57}$$

$$Vo^{+} = 100\left(\frac{50}{100}\right) = 50 \qquad \beta = \frac{2\pi}{3}$$

$$V(z) = 50\ e^{-j\frac{2\pi}{3}z}\left(1 + 0{,}44\ e^{-j\left(\frac{4\pi}{3}z - 0{,}128\right)\pi}\right)$$

b) $V_i = V(0) = 50\left(1 + 0{,}44\ e^{j(-0{,}128\pi)}\right)$

c) $V_L = V_L(3{,}6)$

d) $VSWR = \dfrac{1+\Gamma}{1-\Gamma} = \dfrac{1 + 0{,}44\ e^{j11{,}57}}{1 - 0{,}44\ e^{j11{,}57}}$

e) $P = \dfrac{1}{2}\dfrac{|V_L|^2}{Z_L} R_0$ $\qquad$ P = 0,119 W

SMITH CHART

Transmission-line calculations such as the determination of input impedance, reflection coefficient and load impedance often involve tedious manipulations of complex numbers. This tedium can be alleviated by using a graphical methot of solution. The best known and most widely used graphical chart is the `Smith chart` devised by P.H. Smith in 1939. Smith chart is a graphical plot of normalized resistance and reactance functions in the reflection -coefficient plane. In order to understand how the Smith chart for a lossless transmission line is constructed, let us examine the voltage reflection coefficient of the load impedance.

$$\Gamma = \frac{Z_L - R_0}{Z_L - R_0} = |\Gamma| e^{jQr} \tag{133}$$

Let the load impedance be normalized with respect to the characteristic impedance of the line.

$$z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \tag{134}$$

where and are the normalized resistance and normalized reactance respectively. Equation (133) can be rewritten as

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \tag{135}$$

where ,and are the real and imaginary parts of the voltage reflection coefficient respectively. The inverse relation of Equation (135) is

$$z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma| e^{jQr}}{1-|\Gamma| e^{jQr}} \tag{136}$$

or

$$r + jx = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} \tag{137}$$

Multiplying both the numerator and the denumerator of Equation (137) by the complex conjugate of the denumerator, and separating the real and imaginary parts, we obtain

$$r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \tag{138}$$

and

$$x = \frac{2\Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \tag{139}$$

If equation (138) is plotted in the plane for a given value of, the resulting graph is the locus for this. The locus can be recognized when the equation is rearranged as

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \tag{140}$$

It is the equation for a circle having a radius of 1/(1+r) and centered at (r/(1+r) ,0). Different values of r yield circles of different positions in the reflection coefficient plane. A family of these circles are shown in figure 1. Since only that part of graph lying within the unit circle on the plane is meaningful; everything the outside can be disregarded.

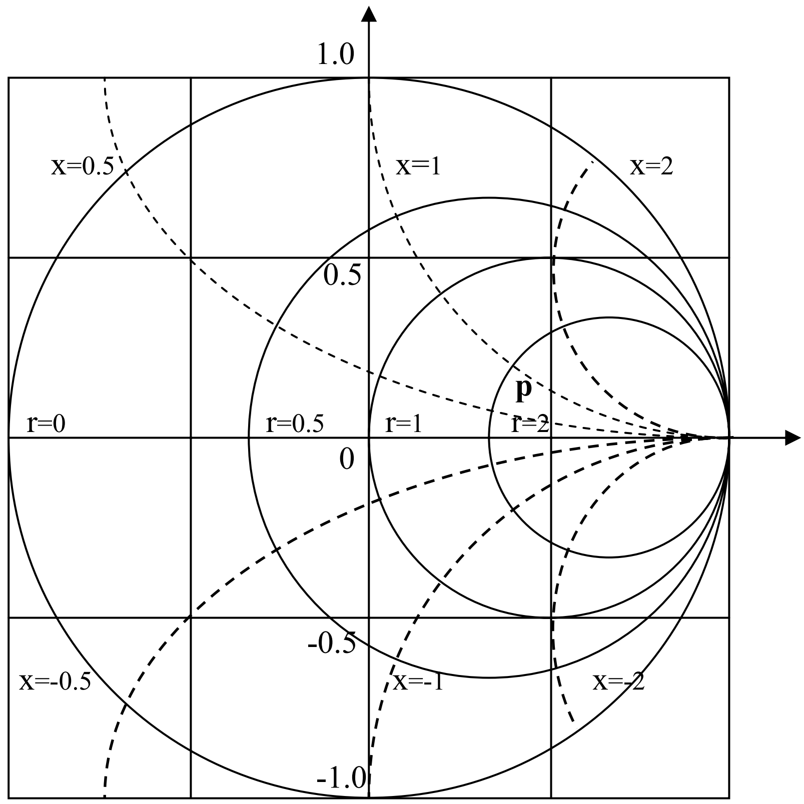


Fig:1 Smith chart with the rectangular coordinates

Several salient properties of the r-circles are noted as follows:

1. The centers of all r-circles lie on the Γ_r – axis.

2. The r = 0 circle, having a unity radius and centered at the origin, is the largest.

3. The r-circles become progressively smaller as *r* increases from 0 toward ∞ , ending at the ($\Gamma_r = 1$, $\Gamma_i = 0$) point.

4. All r-circles pass through the ($\Gamma_r = 1$, $\Gamma_i = 0$) point.

Similarly , (139) may be rearranged as

$$(\Gamma_r - 1)^2 + (\Gamma_i - 1/x)^2 = (1/x)^2 \qquad (141)$$

This is the equation for a circles having radius $1/|x|$ and centered at $\Gamma_r = 1$ and $\Gamma_i = 1/x$.
Different values of x yield circles of different radii with centers at different position on the $\Gamma_r = 1$ line. A family of the portions ıf x-circles lying inside the $|\Gamma| = 1$ boundary are shown in dashed lines in Fig 1. The following is a list of several salient properties of the x-circles.

1. The centers of all x-circles lie on the $\Gamma_r = 1$ line: those for $x > 0$ (inductive reactance) lie above the Γ_r– axis and those for $x < 0$ (capacitive reactance) lie below the Γ_r– axis.
2. The x = o circle becomes the Γ_r– axis.
3. The x-circles become progressively smaller as $|x|$ increases from 0 toward ∞, ending at the ($\Gamma_r = 1$, $\Gamma_i = 0$) point.
4. All x-circles pass though the ($\Gamma_r = 1$, $\Gamma_i = 0$) point.

A smith chart is a chart of r- and x-circles in the $\Gamma_r - \Gamma_i$ plane for $|\Gamma| \leq 1$. It can be proved that the r- and x- circles are everywhere orthogonal to one another. The intersection of an r- and an x-circles defines a point that represents a normalized load impedance $z_L = r + jx$. The actual load impedance is $Z_L = R_0 (r + jx)$. Since a Smith chart plots the normalized impedance, it can be used for calculations concerning a lossless transmission line with an arbitrary characteristic impedance.

As an illustration, point P in Fig. 1. is the intersection of the $r = 1.7$ circle and the $x = 0.6$ circle. Hence it represents $z_L = 1.7 + j0.6$. the point P_{sc} at ($\Gamma_r = -1, \Gamma_i = 0$) corresponds to $r = 0$ and $x = 0$ and, therefore, represent a short-circuit. The point P_{oc} at ($\Gamma_r = 1, \Gamma_i = 0$) corresponds to an infinite impedance and represent an open-circuit.

The Smith chart in Fig.1 marked with Γ_r and Γ_i rectangular coordinates. The Smith chart can be marked with polar coordinates, such that every point in the Γ - plane is specified by a magnitude $|\Gamma|$ and a phase angle θ_Γ. This is illustrated in Fig 2, where several $|\Gamma|$-circles are shown in dotted lines and some θ_Γ -angles are marked around the $|\Gamma| = 1$ circle. The $|\Gamma|$-circles are normally not shown on commercially available Smith charts: but once the point representing a certain $z_L = r + jx$ is located, it is a simple matter to draw a circle centered at the origin thorough the point. The fractional distance from the center to the point (compared with the unity radius to the edge of the chart) is equal to the magnitude $|\Gamma|$ of the load reflection coefficient; and the line to the point makes with the real axis is θ_{Γ}.

$$R_L = \frac{R_O}{S} \qquad (142)$$

Each $|\Gamma|$-circle intersects the real axis at two points. In Fig. 2 we designate the point on positive-real axis (OP_{oc}) as P_M and the point on the negative-real axis (OP_{sc}) as P_m . Since x =0 along the real axis, P_M and P_m both represent situations with a purely resistive load, $Z_L = R_L$. obviously $R_L > R_0$ at P_M, where $r > 1$: and $R_L < R_0$ at P_m, where $r < 1$. We found that $S = R_L/R_0 = r$ for $R_L > R_0$. This relation enables us to say immediately, without using Eq.(142) that the value of the r-circle passing through the point P_M is numerically equal to the standing-wave ratio. Similarly, we conclude from Eq.(142) that the value of the r-circle passing through the point P_m on the negative-real axis is numerically equal to 1/S. For the z_L =_1.7+ j0.6 point, marked P in Fig. 2, we find $|\Gamma| = 1/3$ and $\theta_\Gamma = 28^o$. at P_M, $r = S = 2.0$ these results can be verified analytically.

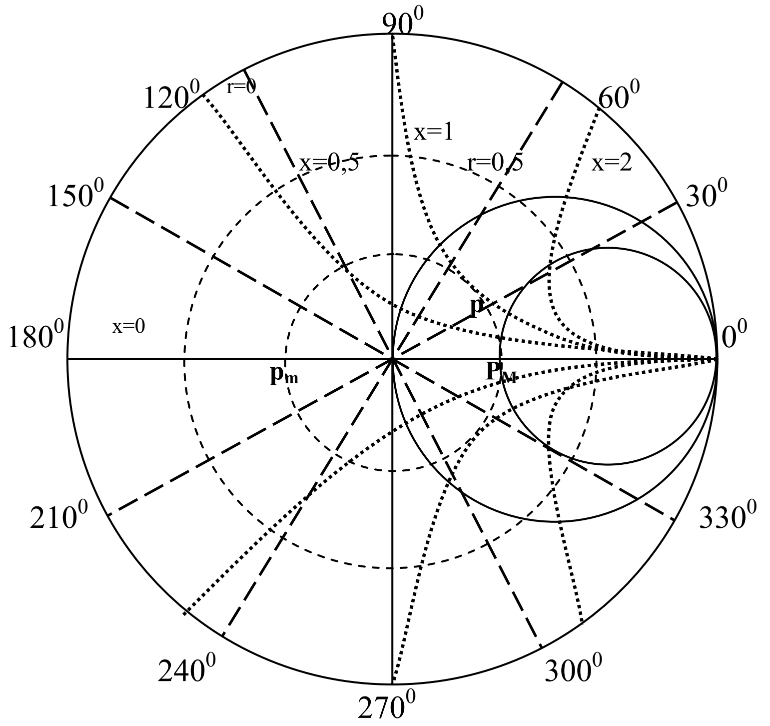


Figure 2

In summary, we note the following:.

1. All $|\Gamma|$-circles are centered at the origin, and their radius vary uniformly from 0 to 1.
2. The angle, measured from the positive real axis, of the line drawn from the origin through the representing z_L equals θ_Γ.
3. The value of the *r*-circle passing through the intersection of the $|\Gamma|$-circle and the positive-real axis equals the standing-wave ratio S.

So far we have based the construction of the Smith chart on the definition of the voltage reflection coefficient of the load impedance. The input impedance looking toward the load at a distance z' from the load is the ratio of V(z') and I(z'). We have, by writing $j\beta$ for γ for a lossless line.

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0\left[\frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}\right] \tag{143}$$

The normalized input impedance is

$$z_i = \frac{Z_i}{Z_0} = \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}} \tag{144}$$

$$= \frac{1+|\Gamma| e^{j\phi}}{1-|\Gamma| e^{j\phi}} \tag{145}$$

where

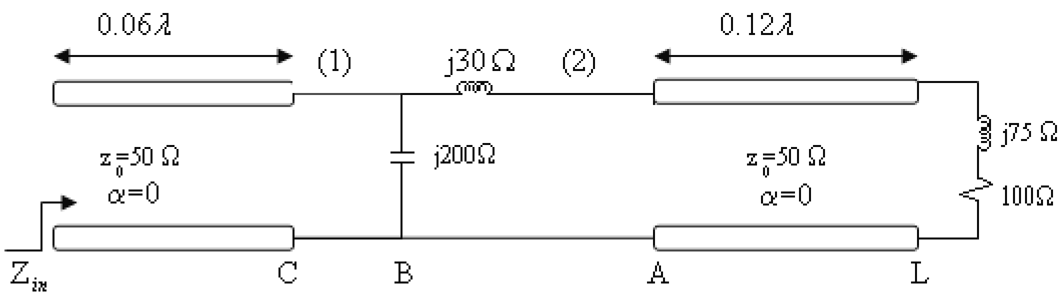
$$\phi = \theta_\Gamma - 2\beta z' \tag{146}$$

We note that Eq.(144) relating z_i and $\Gamma e^{-j2\beta z'} = |\Gamma|\, e^{j\phi}$ is of exactly the same form relating z_L and $\Gamma = |\Gamma|\, e^{j\theta_\Gamma}$. In fact, the latter is a special case of the former for z' =0 ($\phi = \theta_\Gamma$) . The magnitude, $|\Gamma|$, of the reflection coefficient and, therefore, the standing-wave ratio S, are not changed by the additional line length z'. thus just as we can use the Smith chart to find $|\Gamma|$ and θ_Γ for a given z_L at the load, e can keep $|\Gamma|$ constant and subtract (rotate in the clockwise direction) from θ_Γ an angle equal to $2\beta z' = 4\pi z'/\lambda$.

This will locate the point for $|\Gamma| e^{j\phi}$, which determines z_i . Two additional scales in $\Delta z'/\lambda$ are usually provided along the perimeter of the $|\Gamma| = 1$ circle for easy reading of the phase change $2\beta(\Delta z')$ due to a change in line length $\Delta z'$: the outer scale is marked ‘’ wavelength towards generator ‘’ in the clockwise direction (increasing z') ; and the inner scale is marked ‘’ wavelength towards load ‘’ in the counterclockwise direction (decreasing z'). Figure 1.03 is a typical

Smith chart, which is commercially available. It has a complicated appearance, but actually it consists merely of constant-*r* and constant-*x* circles. We note that a change of half-a-wavelength in line length ($\Delta z' = \lambda/2$) corresponds to a $2\beta(\Delta z') = 2\pi$ change in ϕ . A complete revolution around a $|\Gamma|$-circle returns to the same point and results in no change in impedance.
In the following we shall illustrate the use of the Smith chart for solving some typical transmission-line problems by several examples.

SMITH CHART APPLICATION



a) $Z_A, Z_B, Z_C, Z_{in} = ?$

b)Find VSWR, maximum and minimum voltage positions for both lines.

SOLUTION :
Solution is obtained by the Analytical and Graphical methods.

1)Analytical method

$$Z_L \rightarrow \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1} \qquad (z_L = \frac{Z_L}{Z_0})$$

$$z_L = 2 + 1j1.5 \Rightarrow \Gamma_L = \frac{2 + j1.5 - 1}{2 + j1.5 + 1} = \frac{1 + j1.5}{3 + j1.5} = 0.46 + j0.26$$

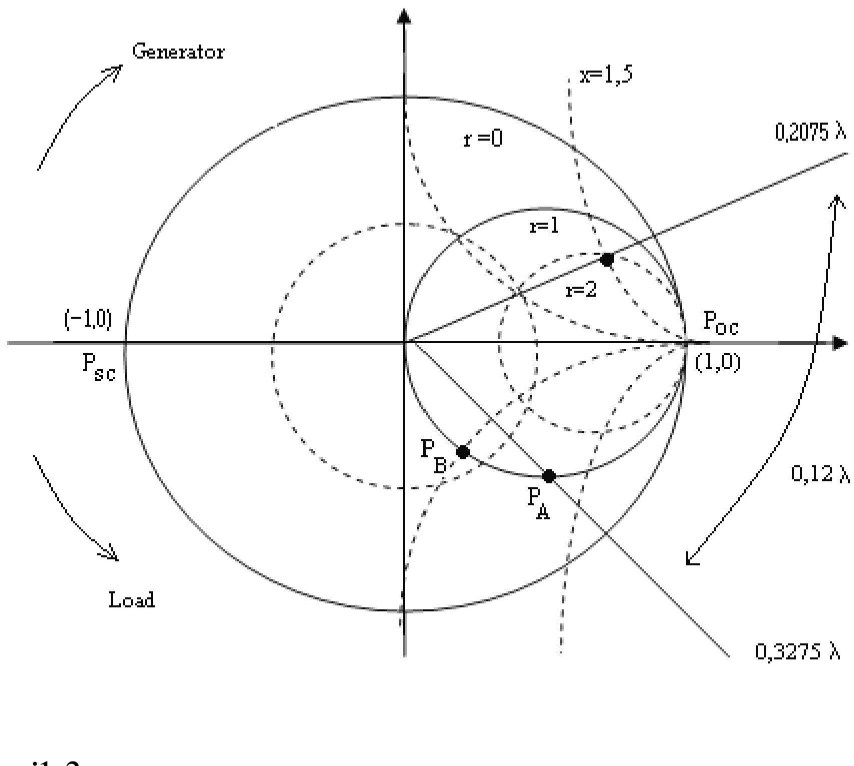
$$Z_L = 2.3 + j1.3$$

$$Z_A = \frac{V_A}{I_A} = Z_0 \frac{1+\Gamma_A}{1-\Gamma_A} = \frac{1+\Gamma_L e^{-j\beta l}}{1-\Gamma_L e^{-j\beta l}} = Z_0 \frac{1+\Gamma_L e^{-j\frac{4\pi}{\lambda}0.12\lambda}}{1-\Gamma_L e^{-j\frac{4\pi}{\lambda}0.12\lambda}} = Z_0 \frac{1+\Gamma_L e^{-j0.48\pi}}{1-\Gamma_L e^{-j0.48\pi}}$$

$$Z_B = Z_A + j30 \qquad Y_B = \frac{1}{Z_B} \qquad Y_C = Y_B + \frac{1}{-j200} \qquad Z_C = \frac{1}{Y_C}$$

$$\Gamma_C = \frac{z_C - 1}{z_C + 1} \Rightarrow Z_{in} = Z_0 \frac{1+\Gamma_C e^{-j\frac{4\pi}{\lambda}0.06\lambda}}{1-\Gamma_C e^{-j\frac{4\pi}{\lambda}0.06\lambda}} = Z_0 \frac{1+\Gamma_C e^{-j0.24\pi}}{1-\Gamma_C e^{-j0.24\pi}}$$

2)Graphical Method



$$P_A : Z_A = 1 - j1.3$$

$$Z_B = 1 - j1.3 + j\frac{30}{50} = 1 - j0.7$$

$P_B : Z_B = 1 - j0.7 \rightarrow \Gamma = 1$, j=0.7

$Y_B \rightarrow$ After taking symmetry of P_B according to the origin Y_B could be found on the graph

$$P_B^{'} : Y_B = 0.67 + j0.47$$

$$Y_C = Y_B + \frac{j}{200}.50 = 0.67 + j0.72$$

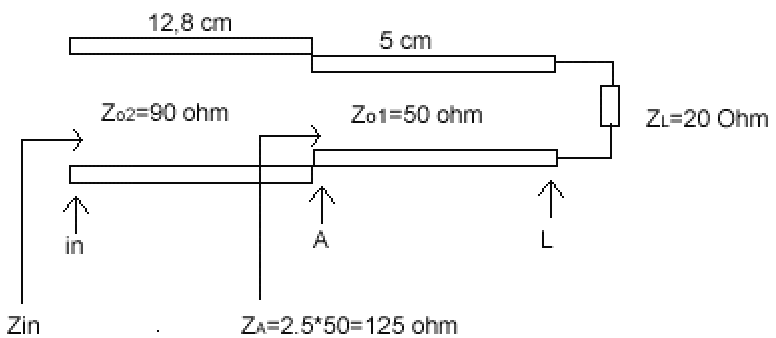
$P_C \rightarrow$ After taking symmetry of Y_C according to the origin , P_C is found which corresponds to the impedance:

$$P_C : Z_C = 0.7 - j0.74$$

$$Z_{in} = z_0.z_{in} = 50(0.45 - j0.38)\Omega$$

Z_{in}=22.5-j19Ω → real input impedance

Example 2



$$\left(\frac{l_1}{\lambda}\right)\lambda = \left(\frac{5cm}{20}\right)\lambda = 0.25\lambda \text{ 1.Line}$$

$$\left(\frac{l_2}{\lambda}\right)\lambda = \left(\frac{12,8cm}{20}\right)\lambda = 0.64\lambda \text{ 2.Line}$$

$$z_L = \frac{Z_L}{Z_{0_1}} = \frac{20}{50} = 0{,}4\Omega \qquad VSWR = \frac{1}{z_L} = 2.5 \qquad R_m = \frac{z_0}{R_s} \qquad |r| = \frac{50}{20} = 2{,}5$$

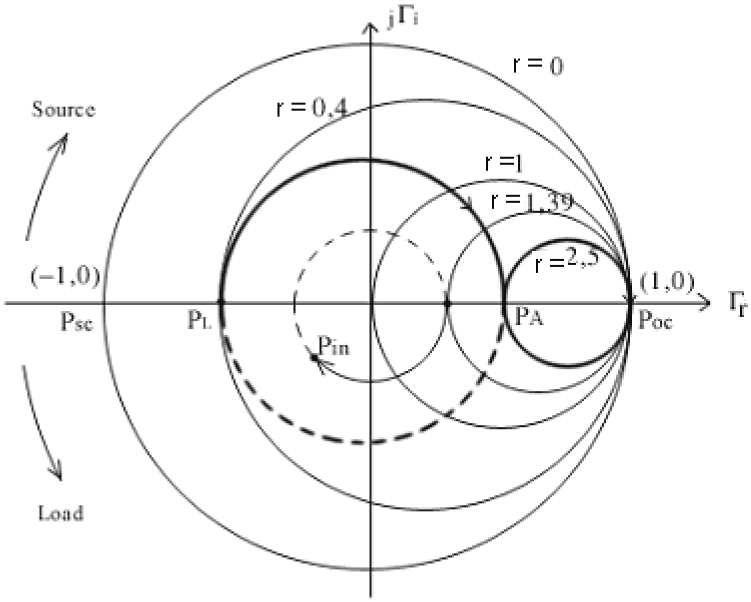
<u>second line</u>

$$Z_A = \frac{125}{90} = 1.39\Omega$$

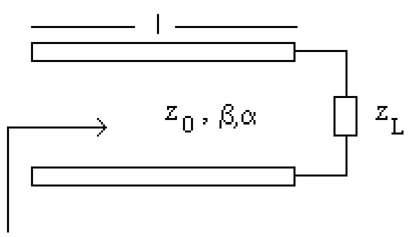
$0.64\lambda = 0.5\lambda + 0{,}14\lambda$ $\qquad$ $P_{in} = 0{,}9 - j0{,}3$

$Z_{in} = z_{02}.z_{in} \quad \Rightarrow \; Z_{in} = 81 - j27\Omega$

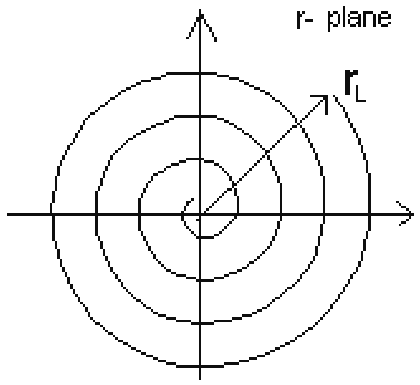
$\Gamma_{in} = 0{,}165.\angle -100^\circ$ $\qquad$ $\Gamma_{in} = \dfrac{Z_{in} - Z_0}{Z_{in} + Z_0}$



Using Smith Chart on Lossy Lines



$$Z_{in} = \frac{Z_{in}}{Z_0} = \frac{1+\Gamma_{in}}{1-\Gamma_{in}} = \frac{1+|\Gamma_L|e^{j\theta_L}e^{-2\alpha d}e^{-j2\beta d}}{1-|\Gamma_L|e^{j\theta_L}e^{-2\alpha d}e^{-j2\beta d}}$$



Position of the Γ_L on the lossy lines (Lowering of the modules because of lossness of the line)

‘

$$\boxed{Z_{in} = \frac{1+|\Gamma_L|e^{-2\alpha d}e^{j(\theta_L-2\beta d)}}{1-|\Gamma_L|e^{-2\alpha d}e^{j(\theta_L-2\beta d)}}}$$

Formula of the normalized input impedance on lossy line

$$Z_{line} = \frac{V_{line}}{I_{line}}$$

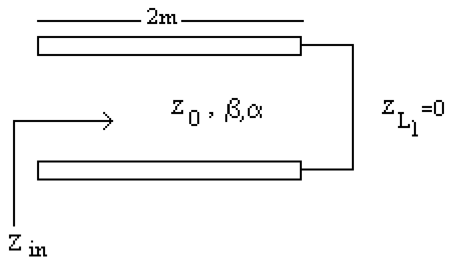
Numerical Application

Z_{L1}=0 , l=2m , Z_0=75Ω , Z_{in}=45+j225Ω

a)α , β =?

b)Z_{L2}=67,5-j45Ω → Z_{in}=?

a)



$$\Gamma_{L1} = \frac{Z_{L1} - Z_0}{Z_{L1} - Z_0} = -1 \Rightarrow \Gamma_L = 1e^{j\pi}$$

$$\Rightarrow \theta_L = \pi rad$$

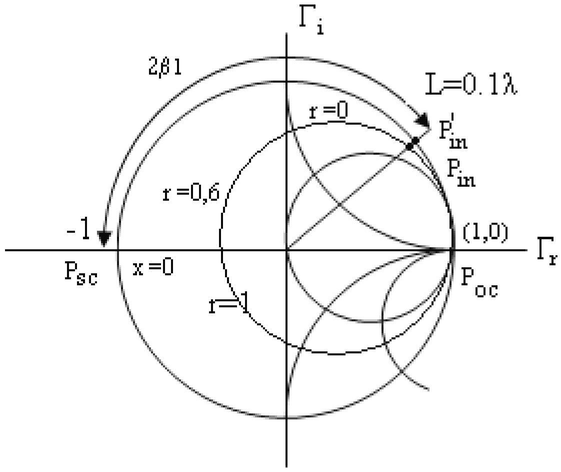
$$z_{in} = \frac{Z_{in}}{Z_0} = \frac{45 + j225}{75} = 0{,}6 + j3$$

$$z_{in} = \frac{1+\Gamma_{in}}{1-\Gamma_{in}} = \frac{1+\Gamma_L e^{-2\gamma(l-z)} e^{-2\alpha d} e^{-j2\beta d}}{1-\Gamma_L e^{-2\gamma(l-z)} e^{-2\alpha d} e^{-j2\beta d}} \quad d = l - z, for z = 0$$

$$z_{in} = \frac{1+e^{j\Pi} e^{-2\alpha l} e^{-j2\beta l}}{1-e^{j\Pi} e^{-2\alpha l} e^{-j2\beta l}} = \frac{1+e^{-2\alpha 2} e^{j(\Pi-2\beta 2)}}{1+e^{-2\alpha 2} e^{j(\Pi-2\beta 2)}} = 0{,}6 + j3$$

If the line is lossless input impedance is purely reactive.Also it could be inductive or capacitive.This condition changes by the lenght of line.

Graphical Solution



α and β are found by using the formulas below

$$\frac{|OP_{in}|}{|OP_{in}'|} = e^{-2\alpha l} \qquad \theta = 2\beta l$$

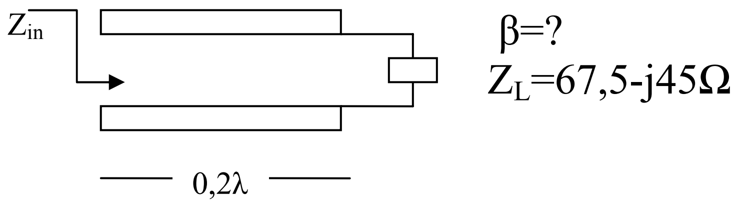
$$\frac{|OP_{in}|}{|OP_{in}'|} = e^{-2\alpha l} = 0,89 \Rightarrow \frac{\ln 0.89}{-2l} = 0,028\,Np/m = \alpha = 0,25 \text{ dB}$$

(1Np=8.69 dB),

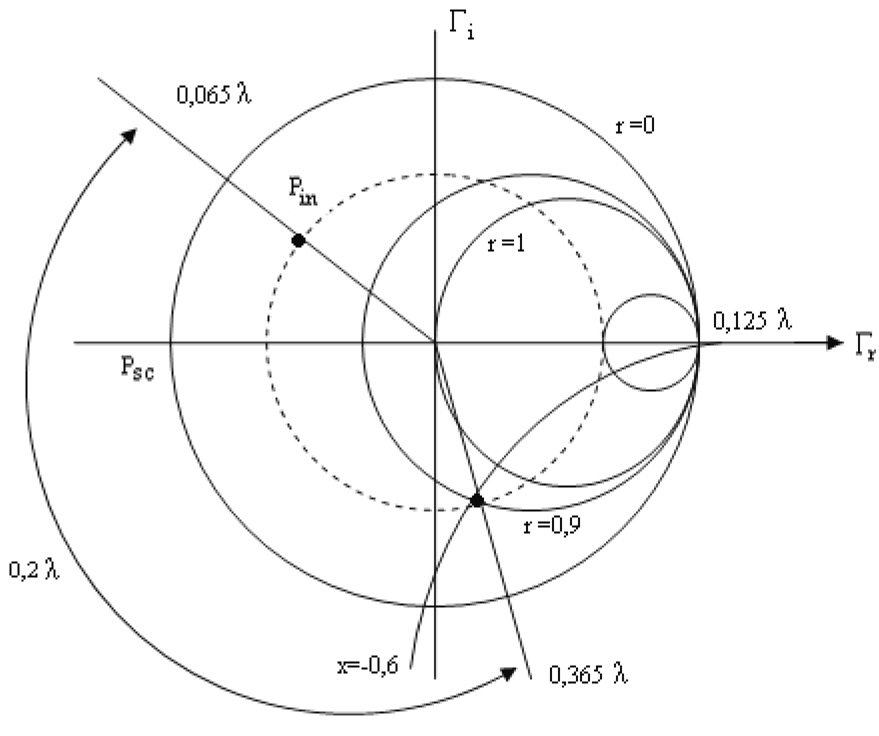
$$Z_{i_1} = 45 + j225\Omega$$

α=0,029 ,
β=0,2π
Z_o=75Ω

b)



$$z_L = \frac{Z_L}{z_0} = 0,9 - j0,6$$

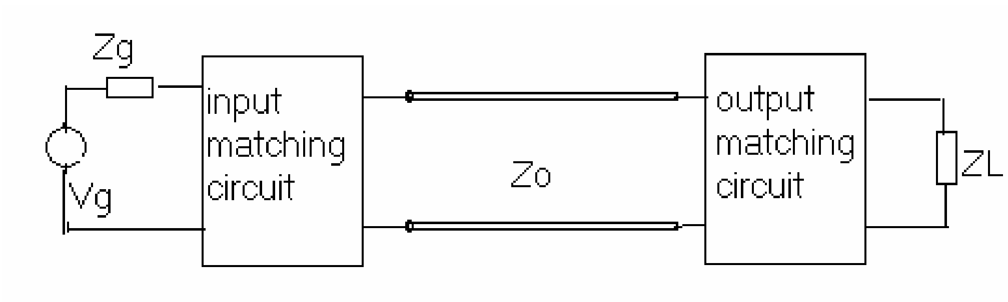


0.2λ towards generator

$$0,365 + 0,2 = 0,565\lambda \equiv 0,065\lambda$$

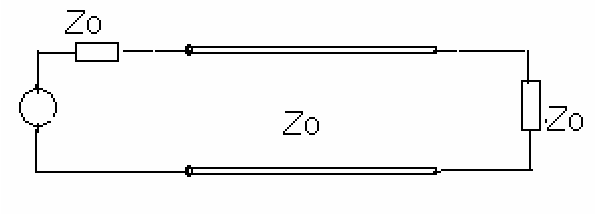
$$Z_{in}=z_0.z_{in}=54,75+j20,25$$

IMPEDANCE MATCHING



Impedance matching is one of the most important subjects of transmission lines. If the characteristic impedance Zo of the line is equal to the load impedance Z_L, the reflection coefficient $\Gamma_L=0$, and the standing wave ratio is unity. When this situation exists, the characteristic impedance of the line and the load impedance are said to be matched, that is, they are equal. In most transmission line applications, it is desirable to match the load impedance to the characteristic impedance of the line in order to reduce reflections standing waves that jeopardize the power-handling capabilities of the line and also distort the information transmitted. Impedance matching is also desirable in order to drive a given load most efficiently (i.e. to deliver maximum load), although maximum efficiency also requires matching the generator to the line at the source end. In the presence of sensitive components (low-noise amplifiers), impedance matching improves the signal-to-noise ratio of the system in other cases generally reduces amplitude and phase errors.

The equivalent circuit is shown below:



$P(z)= P^{+} (1- |\Gamma(z)|2)$ $\Rightarrow$ the power formula

1) $\Gamma g=0 \longrightarrow$ Zg=Zo P+=Pmax (maximum power tranfer)
2) $\Gamma_L=0 \longrightarrow$ Z_L=Zo PL=P+

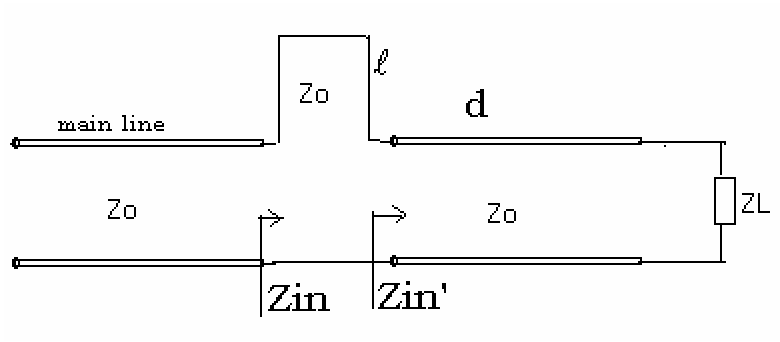
There are different methods of achieving impedance matching:

1-Matching using series or parallel lumped reactive elements
2-Single stub matching (series or shunt)
3-Double stub matching
4-Triple stub matching

SINGLE STUB SERIES IMPEDANCE MATCHING:

At microwave frequencies, it is often impractical or inconvenient to use lumped elements for impedance matching. Instead, we use a common matching technique that uses single open or short-circuited stubs connected either in series or in parallel. In practice, the short-circuited stub is more commonly used for coaxial and wave-guide applications because a short-circuited line is less-sensitive to external influences (such as capacitive coupling and pick-up) and radiates less than an open-circuited line segment. However, for microstrips and striplines, open-circuited stubs are more common in practice because they are easier to fabricate.

The principle of matching with stubs is similar to matching using lumped reactive elements. The only difference is that the matching impedance (Zs) is intruduced by using open or short-citcuited line segments at appropriate length (ℓ).



In the figure above we can see a short-circuited single stub series empedance matching circuit. Here, we will find out appropriate ℓ and d lengths that the input empedance of the matching circuit becomes Zo (Zin=Zo).

As we study at normalized dimensions, following equations can be found:

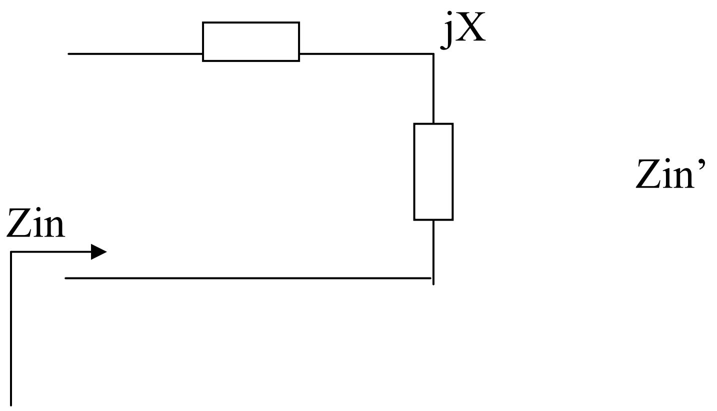
$z_{in} = \overline{Z}in = Zin/Zo$ and $z_L = \overline{Z}_L$

$z_{in}' = (z_L + j.\tan\beta d) / (1 + j.z_L.\tan\beta d) = 1 + j\ \overline{X}in'$

This is the input empedance that is observed from right side of the stub!

$Re\{z_{in}'\}=1$ $Im\{z_{in}'\}= \overline{X}in'$

The equivalence of the matching circuit is like this:



$z_{in} = z_{in}' + jx = 1 = 1 + j0$

$z_{in} = 1 + \overline{j}Xin' + jx = 1 + j0$

$Xin' + X = 0$

$X = -Xin'$

So, chosen ℓ and d lengths must supply these equations.

- Let's think about pure resistive load empedance ($Z_L = R$, $z_L = \overline{R} = r$)

If tanβd=t, then

$zin' = 1 + j\ \overline{X}in' = (r + j.t) / (1 + j.r.t)$

$(1 + j\ \overline{X}in').(1 + j.r.t) = (r + j.t)$

Imaginary and real parts of both sides will be equal:

$1 - \overline{X}in'.r.t = r$

$j.(\overline{X}in' + r.t) = j.t \quad \Rightarrow \quad \overline{X}in' = (1 - r).t$

$t = (1 - r) / (1 - r).r.t \quad \Rightarrow \quad t^2 = \tan^2\beta d = 1/r$

$\tan^2\beta d = (1 - \cos^2\beta d) / \cos^2\beta d = 1/r$

By this equation, d can be found like this:

$d = (\lambda/4\pi) . \arccos[(r - 1) / (r + 1)]$

And ℓ can be found as below:

$-j\ \overline{X}in' = -j\cot\beta\ell \qquad \Rightarrow \qquad \ell = (\lambda/2\pi). \text{Arctan}(\sqrt{r} / 1 - r)$

- If the load empedance is not pure resistive (zL = rL + xL) :

Then we look at the max. points of the wave:

r_{max} = VSWR = $(1 - |\Gamma_L|) / (1 + |\Gamma_L|) = \bar{R}$

So, new formulas of d and ℓ are:

$d' = (\lambda/4\pi) \cdot \arccos[(VSWR - 1) / (VSWR + 1)]$ $d = d' + dmax$

$\ell = (\lambda/2\pi) \cdot \text{Arctan} (\sqrt{VSWR} / 1 - VSWR)$

GRAPHICAL SOLUTIONS:

Impedance matching problems can be solved easily using the Smith Chart. Let's look at an antenna matching example:

- To consider stub matching it helps to have a practical example. Here, we study a load

formed by an antenna which is being used away from its design frequency. The method is not restricted to antenna loads.

For a 1 metre long dipole antenna at 120 MHz, the load impedance is 44.8 ohms - j 107 ohms. The normalised empedance is 0.597 - j 1.43 with respect to the 75 ohm coaxial line. We shall determine the position and length of a series stub which will match this antenna to the transmission line.

If we look at the **SMITH Chart** we find a circle of constant real normalised impedance r=1 which goes through the open circuit point and the centre of the chart. In our example in the next picture, this circle is drawn in red. If you plot any arbitrary normalised impedance on the SMITH chart, and follow round clockwise at constant radius, from the centre of the SMITH chart, towards the generator (along the green line in the example), you *must* cross the r=1 circle somewhere. This transformation at constant radius represents motion along the transmission line towards the generator. (One complete circuit of the SMITH chart represents a travel of one half wavelength towards the generator.) At this intersection point the generalised arbitrary load impedance r + jx has transformed to **(1+jx')**, so, at least the real part of the impedance equals the characteristic impedance of the line. Matching has not yet been achieved because of the residual reactance x' which must be tuned out with the stub. Note that x' is different from x in general. For each transformation around the SMITH chart, representing travel one half wavelength towards the transmitter, there are two intersections with the r=1 circle. Stubs may be placed at either of these points.

At the transformed (see figure –1) intersection point (red and green circles) the line is cut and a pure reactance -jx' is added. This is done by creating this reactance -jx' using a series-connected lossless stub. Now, the total impedance looking into the sum of the line impedance (which is 1+jx') and -jx' is therefore (1+jx') -jx' = 1 and the line is matched.

Again, one looks at the SMITH chart and finds the outer circle where the modulus of the reflection coefficient is unity. On this circle are the SHORT and OPEN points, and all values of positive (top half of the SMITH chart) and negative (bottom half of the SMITH chart) reactance. The resistance is zero everywhere. It has to be zero, as a lossless transmission line with load infinity ohms (open) or zero ohms (short) has no mechanism for absorbing power. To generate a specified reactance, start at a short circuit (or maybe an open circuit) and follow the rim of the SMITH chart clockwise around towards the generator until the desired reactance is obtained. Cut the stub this number of wavelengths long.

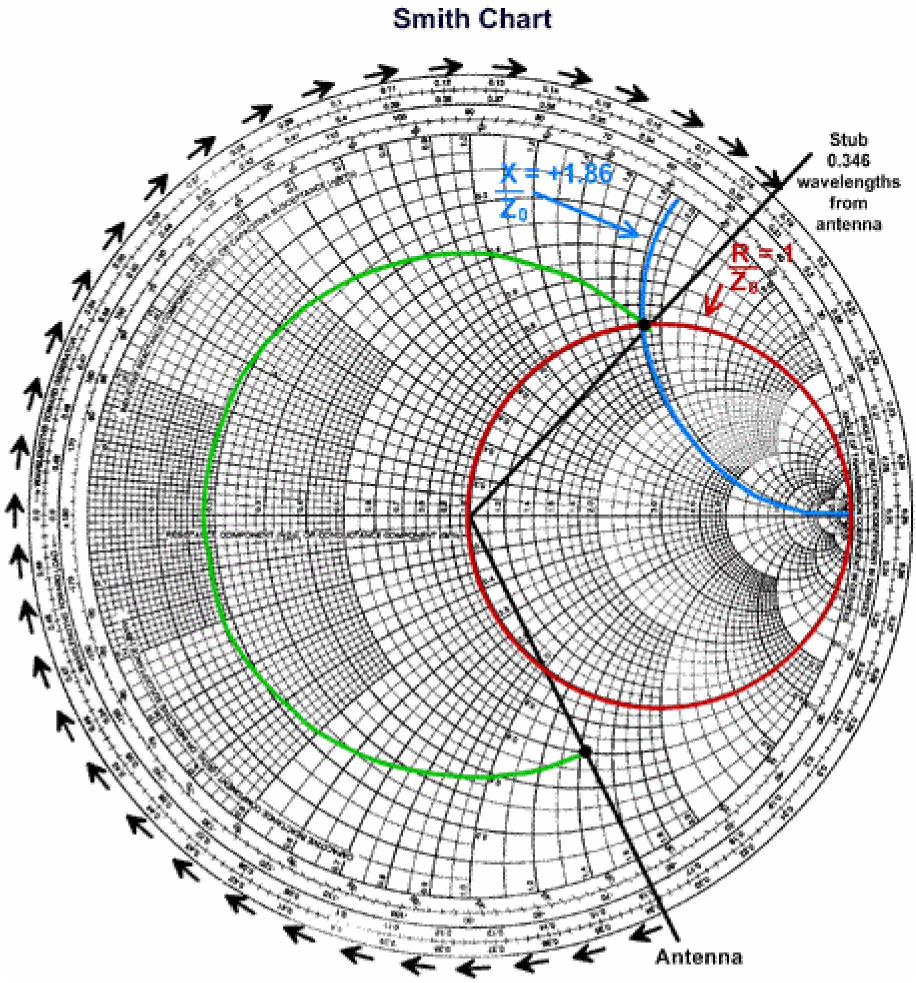


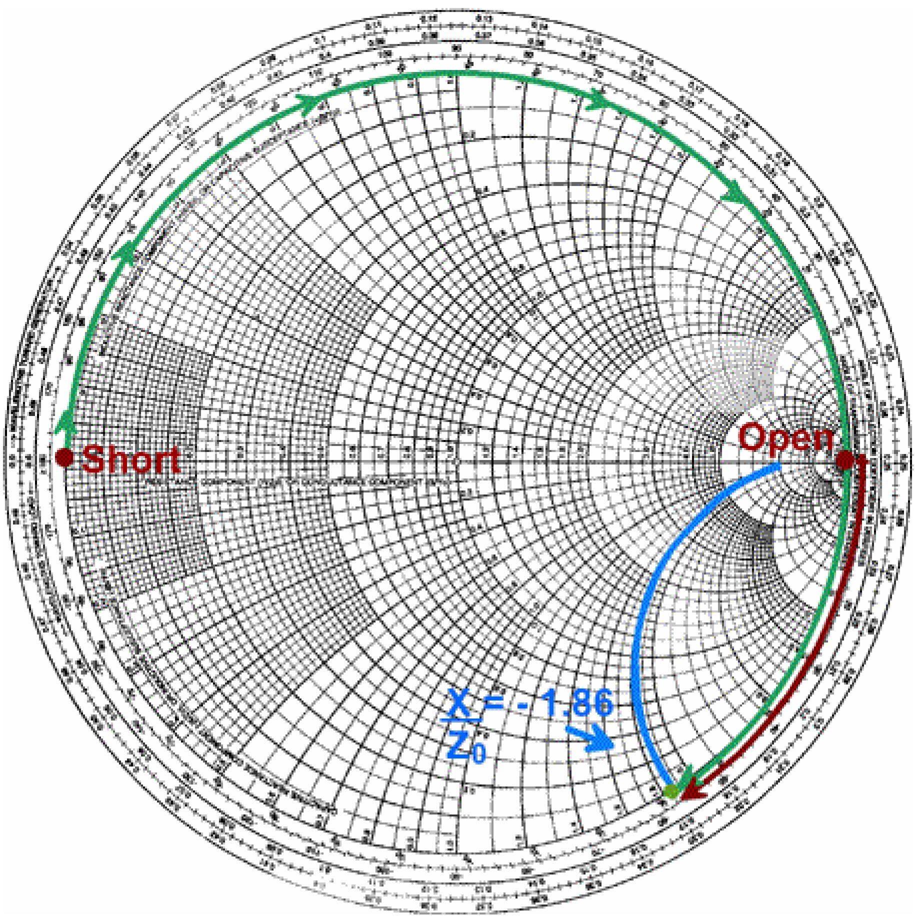
FIG –1

In our example, the SMITH chart construction to find the stub length is shown in the next picture.

From the blue arc in the previous picture we see that the reactance at the r=1 intersection point is +j1.86, so to cancel this out we must add a series stub having reactance -j1.86. In the next figure we plot the blue arc -j1.86 and, starting from the short circuit (r = x = 0) we follow the green line around a distance of 0.328 wavelengths clockwise towards the generator, to generate this value of reactance. If we had started from an open circuit we would only travel a distance (0.328 - 0.250) =

0.078 wavelengths to generate this reactance. This open circuit stub is represented by the red arc.

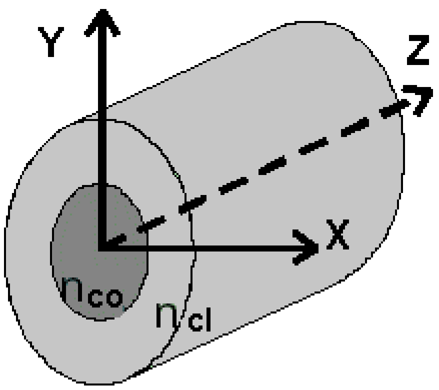
The practical details of the series stub match are shown in third figure, where we display the physical lengths in centimetres, assuming a wave velocity on the coax (which we need to know to do this calculation) of 2x10^8 metres per second. This data is supplied by the cable manufacturer. The wave velocity and the frequency (120 MHz) allows us to calculate the wavelength in metres, and thus we can translate the "electrical lengths" from the SMITH chart into physical lengths of line.



$\ell = 0.174\lambda$ and the stub position from load will be $d = 0.47\lambda$.

THE ANALYSIS OF THE GENERAL CYLINDRICAL TRANSMISSION LINES

We consider a cylindrical waveguide of arbitrary cross-sectional shape. The long axis of the waveguide is along the z-direction. The walls of the waveguide are perfect conductors, and the material within the waveguide is characterized by ε, μ .



$$\nabla \times \vec{E} = -jw\mu\vec{H} \qquad \textbf{(Faraday)}$$

$$\nabla \times \vec{H} = (\sigma + jw\varepsilon)\vec{E} + Ju \qquad \textbf{(Ampere)}$$

$$\nabla \vec{E} = \frac{\rho}{\varepsilon} \qquad \textbf{(Gauss)}$$

$$\nabla \vec{H} = 0 \qquad \textbf{(Gauss)}$$

$$\{\nabla^2 + k^2\} \begin{matrix} \vec{E}(x,y,z) \\ \vec{H}(x,y,z) \\ \vec{D}(x,y,z) \\ \vec{B}(x,y,z) \end{matrix} = 0 \qquad \underline{\textbf{Helmholtz Equation}}$$

{Ju=0,ρ=0}

The Helmholtz equation is a seperatable linear differential equation. So ;

$$\{\nabla^2 + k^2\} \begin{matrix} \overrightarrow{E_X}(x,y,z) \\ \overrightarrow{E_Y}(x,y,z) \\ \overrightarrow{E_Z}(x,y,z) \end{matrix} = 0$$

General Cylindrical Transmission System :

The equation of a wave propogating along the z- axis :

$$\vec{E}(x,y,z) = \overrightarrow{E_t}(x,y,z) + \overrightarrow{E_Z}(x,y,z) = \vec{e}(x,y)e^{\pm j\mathrm{B}z} + \overrightarrow{e_z}(x,y)e^{\pm j\mathrm{B}z}$$

$$\vec{H}(x,y,z) = \overrightarrow{H_t}(x,y,z) + \overrightarrow{H_Z}(x,y,z) = \vec{h}(x,y)e^{\pm j\mathrm{B}z} + \overrightarrow{h_z}(x,y)e^{\pm j\mathrm{B}z}$$

As all the EM wave components have to prove the Maxwell Equations, we can analyse these equations for the general cylindrical transmission lines.

Defining the transverse gradient ∇_τ ,

$$\nabla_t = \frac{\partial}{\partial x}\overrightarrow{a_x} + \frac{\partial}{\partial x}\overrightarrow{a_y}$$

We have ;

$$\nabla \times \vec{E} = (\nabla_t + \frac{\partial}{\partial x}\overrightarrow{a_z}) \times \vec{E} = (\nabla_t - j\beta\overrightarrow{a_z}) \times (\vec{e} + \overrightarrow{e_z})e^{-j\beta z}$$

$$= -\mathrm{jw}\mu_0 \; (\vec{h} + \overrightarrow{h_z})e^{-j\beta z} \; (\vec{h} + \overrightarrow{h_z})e^{-j\beta z}$$

$$\underbrace{\nabla_t \times \vec{e}}_{\substack{\text{longitudinally} \\ \text{component}}} \underbrace{- j\beta\overrightarrow{a_z} \times \vec{e}}_{\substack{\text{breadthways} \\ \text{component}}} + \underbrace{\nabla_t \times \vec{e}_z}_{\nabla_t \times \overrightarrow{a_z e_z} = -\overrightarrow{a_z} \times \nabla_t \overrightarrow{e_z}} \underbrace{- j\beta\overrightarrow{a_z} \times \overrightarrow{e_z}}_{0} = -j\omega\mu_0(\vec{h} + \overrightarrow{h_z})e^{j\beta z}$$

Faraday's Law :

$$\nabla_t \times \vec{e} = -j\omega\mu_0 \vec{h}_z \qquad (1)$$

$$-\overrightarrow{a_z} \times \nabla_t \vec{e}_z - j\beta \overrightarrow{a_z} \times \vec{e} = -j\omega\mu_0 \vec{h} \tag{2}$$

Ampere's Law :

$$\nabla_t \times \vec{h} = -j\omega\varepsilon \vec{e}_z$$

$$\vec{a}_z \times \nabla_t \vec{h}_z + j\beta \vec{a}_z \times \vec{h} = -j\omega\varepsilon \vec{e}$$

To analyze the general cylindrical transmission lines, first we have to obtain $\vec{e}$ and $\vec{h}$ as the parameter of e_z and h_z .

$$\vec{e} = \overrightarrow{g}(e_z, h_z)$$

$$\vec{h} = \overrightarrow{f}(e_z, h_z)$$

Second we have to solve the Helmholtz equation in V domain to obtain $e_z(x,y)$ and $h_z(x,y)$ and finally assign all the EM components in V domain.

If we multiply Eq-2 by $-j\beta\vec{a}_z$ vectorally, we obtain;

$$-j\beta[\vec{a}_z \times (-\vec{a}_z \times \nabla_t \vec{e}_z) - j\beta \vec{a}_z \times (\vec{a}_z \times \vec{e})] = -j\beta(-j\omega\mu)\vec{a}_z$$

$$-j\beta(\vec{a}_z \nabla_t \vec{e}_z)(-\vec{a}_z) + j\beta(\vec{a}_z(-\vec{a}_z))\nabla_t \vec{e}_z$$

$$\boxed{\begin{aligned} (k^2 - \beta^2)\vec{e} &= j\omega\mu_0 \vec{a}_z \times \nabla_t \vec{h}_z - j\beta(\nabla_t \vec{e}_z) \\ (k^2 - \beta^2)\vec{h} &= -j\omega\varepsilon \vec{a}_z \times \nabla_t \vec{e}_z - j\beta(\nabla_t \vec{h}_z) \end{aligned}} \qquad k = \omega\sqrt{\mu_0 \varepsilon}$$

According to these equations, we can seperate EM waves propagating along z-direction in cylindrical transmission lines into four groups:

1) **TE (transverse electric) Waves :** Ez = 0, Hz ≠0
2) **TM (transverse magnetic) Waves :** Hz = 0, Ez ≠0
3) **TEM (transverse electromagnetic) Waves :** Hz = 0, Ez ≠0⇒In this condition β=±k
4) **Hybrid**⇒ $E_z \neq 0$, $H_z \neq 0$

$$\overrightarrow{\mathrm{B}_{\mathrm{t}}} = \frac{j\omega\mu\varepsilon.\overrightarrow{e_z} \times \overrightarrow{\nabla_t} E_z \mp jk\overrightarrow{\nabla_t} B_z}{{k_0}^2 - k^2} \tag{3}$$

$$\{{\nabla_t}^2 + (\mu\varepsilon\omega^2 - k^2\} \begin{matrix} \vec{E}_z \\ \vec{B}_z \end{matrix} = 0 \tag{4}$$

TE waves are sometimes called H-waves and TM waves are sometimes called E-waves, where the E-wave and H-wave notation refers to the field that has a z-component. It is important to realize that TE and TM modes are independent solutions, i.e., they independently satisfy Eq (4) and the boundary conditions at the walls. (A solution where both *Ez* ≠0 and *Bz* ≠0 would not be an additional independent solution, but rather, if it existed it could be constructed from a superposition of degenerate TE and TM modes.However as we shall now see, the fields for TE and TM modes satisfy different boundary conditions. Consequently, they will not be degenerate.)

For TM waves, the boundary condition that the tangential component of E vanishes at the walls means that *Ez* vanishes at the walls. This single BC uniquelydetermines the solution of Eq (4) for TM waves. Therefore, it is unnecessary in the case of TM waves to impose the other boundary condition at the walls, namely, that the normal component of the magnetic field ($\vec{e_n} \cdot \vec{B_t}$ in this case) vanishes there. The lattercondition must be automatically contained in Eq (3) for $\vec{B_t}$ when it is applied to TM waves by setting *Bz* = 0 in the RHS of that equation. To see this, note that the only component of $\vec{\nabla_t E_z}$ that is relevant for finding the normal component of B_t from Eq (3) is the gradient of *Ez* with respect to the coordinate along the boundary, and this vanishes since E_z is constant there (actually, $E_z = 0$ at the walls).

For TE waves there is no E_z , so to solve Eq (4) we use the BC that the normal component of B vanishes at the walls. The latter BC turns out to be equivalent to the condition that the normal derivative of B_z vanishes at the walls. To see this, calculate $\vec{e_n} \cdot \vec{B_t}$ using Eq (3) for B_t . Noting that $E_z = 0$ for TE waves, we find that $\vec{e_n} \cdot \vec{B_t}$ is proportional to $\vec{n} \cdot \nabla_t \overrightarrow{B_z}$, which is identical to the normal derivative *∂Bz/ ∂n*. Thus *∂Bz/ ∂n* vanishes at the walls for a TE wave. No other boundary condition is needed to obtain a unique solution of Eq (4) for TE waves. Therefore, the other boundary condition, namely that the tangential component of $\overrightarrow{E_t}$ vanishes at the walls, must be 6 automatically satisfied by this solution for TE waves. (This is easily shown by an argument analogous to that given in the previous paragraph for the case of TM waves.)

Maxwell Equations in Divergiance Form

$$\nabla \vec{B} = \nabla \mu \overrightarrow{H} = 0 \qquad \nabla \vec{H} = 0$$

$$(\nabla_t - j\beta \overrightarrow{a_z})(\vec{h} + \overrightarrow{h_z}) e^{-j\beta z} = 0$$

$$\nabla_t \vec{h} - j\beta \overrightarrow{h_z} = 0 \qquad \underline{\nabla_t \vec{h} = j\beta h_z}$$

$$\nabla \overrightarrow{D} = 0 \qquad \underline{\nabla_t \vec{e} = -j\beta e_z}$$

Obtaining $E_z(x,y)$ ve $H_z(x,y)$:

$$\left\{\nabla^2 + k^2\right\} \begin{matrix} \vec{E} \\ \vec{H} \\ \vec{B} \\ \vec{D} \end{matrix} = 0 \qquad \Rightarrow \qquad \left\{\nabla^2 + k^2\right\} \frac{\overrightarrow{E_z}}{\overrightarrow{H_z}} = 0 \qquad (5)$$

$$\nabla^2 = \nabla.\nabla = (\nabla_t - j\beta\overrightarrow{a_z})(\nabla_t - j\beta\overrightarrow{a_z})$$

$$\nabla^2 = \nabla_t^{\ 2} - \beta^2 \quad (6)$$

$$\nabla_t = \frac{\partial}{\partial x}\overrightarrow{a_z} + \frac{\partial}{\partial y}\overrightarrow{a_y} \qquad \nabla_t^{\ 2} = \nabla_t.\nabla_t = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(6) $\Rightarrow$ (5) we obtain;

$$\left\{\nabla^2 + k^2\right\} \frac{\overrightarrow{E_z}(x,y,z)}{\overrightarrow{H_z}(x,y,z)} = 0 \qquad \Rightarrow \qquad \frac{\partial^2}{\partial x^2}e_z + \frac{\partial^2}{\partial y^2}e_z + \underbrace{(k^2 - \beta^2)}_{h^2 = k^2 - \beta^2}e_z = 0 \quad ;$$

The solution of this differential equation at E_z=0, gives $E_z(x,y)$.

$$h^2 = k^2 - \beta^2 \quad \Rightarrow \quad \text{Characteristic value}$$

$$k^2 = \left(\frac{\omega}{U_\varepsilon}\right) \quad , \quad U_\varepsilon = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{C}{\varepsilon_r}$$

"h" is the function of the problem's geometry and takes discrete values. We can obtain "h" from the solution of the Helmholtz equation for geometry of the problem. So we obtain;

$$\beta = \mp\sqrt{k^2 - h^2} \qquad \{\beta \in R \ \text{ and } \ k^2 > h^2\}$$

For propogation of EM waves β must be a member R .Using $\beta = \mp\sqrt{k^2 - h^2}$ equation we can analyze the propogation for different conditions of β.

1) For β=0 k=k_{cutoff}=h , $k_C = \dfrac{\omega_C}{U_\varepsilon} = h$

$\omega_c = h\,U_\varepsilon$ $\Rightarrow$ $f_C = \dfrac{h}{2\pi\sqrt{\mu_0\varepsilon}} = \dfrac{h.c}{2\pi\sqrt{\varepsilon_r}}$

If the EM wave frequency is equal to cutoff frequency , then β = 0 . So no propogation is available.

2) For k > h ⇔ k > k_c ⇔ f < f_c

$k^2 - h^2 > 0 \Rightarrow \beta = k\sqrt{1 - \left(\dfrac{f_C}{f}\right)^2}$

$$\Rightarrow \frac{h}{k} = \frac{k_C}{k} = \frac{\omega_C}{U_\varepsilon} = \frac{U_\varepsilon}{\omega} = \frac{\omega_C}{\omega}$$

3) For k < h ⇔ f < f_c

$$\Rightarrow \beta = \mp\sqrt{k^2 - h^2} = \mp\sqrt{-h^2(1 - \frac{k^2}{h^2})}$$

β is imaginer , and causes attenuation.

Summary : General cylindrical wave guides have cut off characteristic.

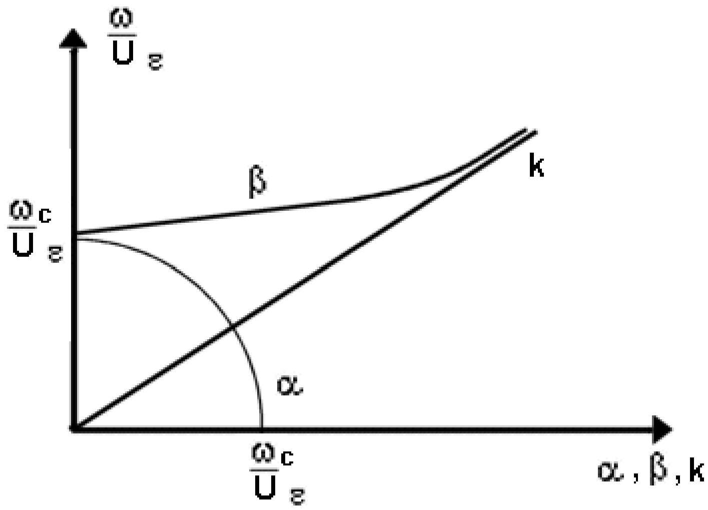
If f = f_c cut off

If f > f_c propogation

If f < f_c attenuation

For $\omega > \omega_c$:

$$\left(\frac{\omega}{U_\varepsilon}\right)^2 - \beta^2 = \left(\frac{\omega_C}{U_\varepsilon}\right)^2$$

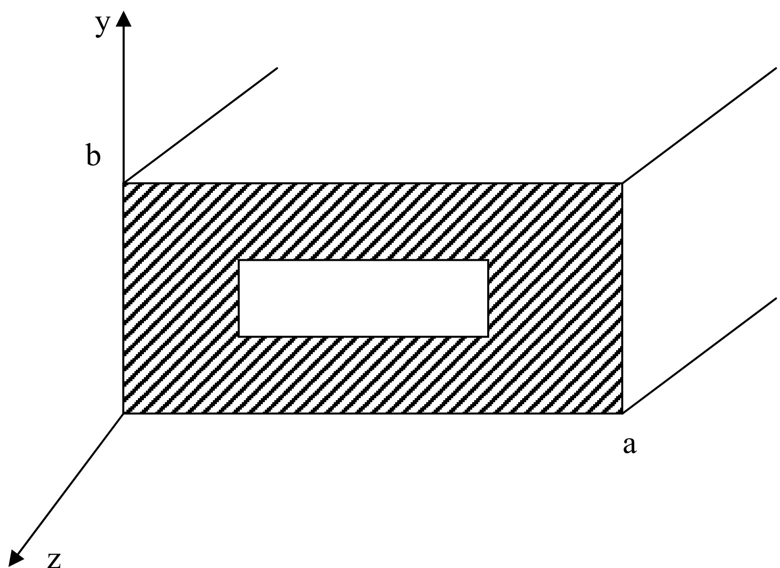


For $\omega < \omega_c$:

$$\left(\frac{\omega}{U_\varepsilon}\right)^2 + \alpha^2 = \left(\frac{\omega_C}{U_\varepsilon}\right)^2$$

RECTANGULAR WAVEGUIDES

The solution of the EM waves propagating in the $\pm z$ direction in the section σ in the systems with only one conductor, the TEM mode cannot exist.



First we must find ez and hz

TE WAVES $\Leftrightarrow \varepsilon_z=0$ $h_z \neq 0$

$\{\nabla t^2+RC^2\}$ hz=0 $\qquad kc^2=k^2-\beta^2=h^2$

$$\frac{\partial^2}{\partial X^2} hz + \frac{\partial^2}{\partial y^2} hz + kc^2 hz = 0 \qquad (1)$$

hz(x,y)= f(x) . g(y) (2)

If we put (2) into (1) and divide with f.g

$$\underbrace{\frac{1}{f}\frac{d^2 f}{dx^2}} + \underbrace{\frac{1}{g}\frac{d^2 f}{dy^2}} + kc^2 = 0 \qquad (3)$$

Only the function of x only the function of y

$-kx^2 - ky^2 + kc^2 = 0$

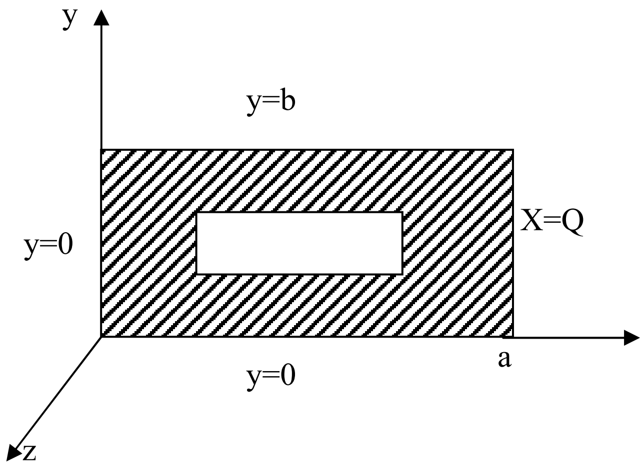
$$\frac{1}{f}\frac{d^2 f}{dx^2} + -kx^2 \Rightarrow \frac{d^2 f}{dx^2} + kx^2 f = 0 \qquad (4.1)$$

$$\frac{1}{g}\frac{d^2g}{dy^2} = -ky^2 \;\Rightarrow\; \frac{d^2g}{dy^2} + ky^2 g = 0 \qquad (4.2)$$

$$kx^2 + ky^2 = kc^2 \qquad (4.3)$$

- From (4.1) $f(x,y)= A_1\cos kx.x + A_2\sin kx.x$ (5.1)
- From (4.2) $g(x,y)= B_1\cos ky.y + B_2 \sin ky.y$ (5.1)

BOUNDARY CONDITIONS



$$\left.\frac{\partial hz}{\partial n}\right|_{boudary} = 0$$

$$\left.\frac{\partial h}{\partial x}\right|_{\substack{X=0\\X=a}} = 0 \qquad \left.\frac{\partial h}{\partial y}\right|_{\substack{Y=0\\Y=b}} = 0 \qquad (16)$$

$$\frac{\partial f}{\partial x} = -KxA_1 \sin kx.x + kxA_2 \cos A_2.\cos kx.x \;\Big]_{\substack{x=0\\x=a}} = 0$$

for x=0 $A_2=0$ (7.1)

for x=a $-kxA_1 \sin kx\, a=0$

$kx.a=m\pi$ m= 0,1 ...

$kx=\frac{m\pi}{a}$ m= 0,1 ..... (7.2)

$$\left.\frac{\partial hz}{\partial y}\right|_{\substack{Y=0\\Y=b}} = 0$$

$\frac{\partial g}{\partial x}$ = --B_1 ky.sin ky.y +B_2ky cos k.y = 0

For y=0 $B_2=0$

(7.3)

For y= b $-B_1 ky \sin ky.b=0$

$ky.b= n\pi$

$ky=\frac{n\pi}{b}$ n= 0,1... (7.4)

Thus,

h(x,y) = f(x) . g(y)

$hz(x,y)= Hmn . \cos \frac{m\pi x}{a} . \cos \frac{n\pi y}{b}$ (8.1)

$$kc^2 = kx^2+ky^2 = kmn^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$Hmn \underset{=}{\Delta} A_1B_1$ (8.2)

$Wc = wmn = kc\, U\in$

$$fc=fmn= \frac{U\in}{2\pi}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

There is ∞ TE modes and all of them have different cut off frequency. There is not EM power of the waves propagating in the ±z direction which belong to

f∞

f= fmn Temn mode status $\beta mn=0$

(9.1)

f>fmn

$$\Gamma\, mn = j\beta mn = j\,(k^2-k^2mn)^{1/2} = j\left[\left(\frac{w}{U\in}\right)^2-\left(\frac{wmn}{U\in}\right)^2\right]^{1/2}$$

$$\Gamma\, mn = j\left[\left(\frac{2\pi}{\lambda}\right)^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]^{1/2} \quad m=0,1..,$$

$$n=0,1...$$

$f < fmn \Rightarrow \Gamma\, mn = \alpha\, mn = (kmn^2-k^2)^{1/2}$ (9.3)

For TMmn hz=0, ez ≠0 $(\nabla t^2 + kc^2)\, ez=0$ are propagation parameters

$TMnn = ez\,(x,y) = Emn \sin \frac{m\pi x}{a}.\sin\frac{n\pi}{b}y$

TMnn = hz (x,y) = Hmn cos $\frac{m\pi x}{a}.\cos\frac{n\pi}{b}y$

$$fmn = \frac{U \in}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad m = 0,1\ldots$$

$$n = 0,1.., \ a > b$$

Thre is ∞x∞ number of TE_{mn} and TM_{mn} modes

$$TE10 = f10 = \frac{U \in}{2a} \qquad \begin{aligned} & f_{10} < f < f_{20} \\ & \frac{U \in}{2a} < f < \frac{U \in}{a} \end{aligned}$$

One mode frequance band

In practice the circular waveguides are mostly used in dominant mode.

In this way one mode propagation is provided.

$T\varepsilon_{01} \rightarrow TM_{11}$

$T\varepsilon_{20} \rightarrow TM_{11}$

The Lowest Cut Off Frequancy is Te_{10}

$TE_{10} \rightarrow f_{10} = \frac{U\varepsilon}{2a}$ The Lowest Cut Off Frequency

$\frac{U \in}{2a} < f < \frac{U \in}{a}$ Allowable Operating Frequency Range

TE_{10} mode is the dominant mode for rectangular waveguides. (a>b)

$$f_{10} = \frac{U \in}{2a} \rightarrow TE_{10} \rightarrow$$ The lowest cut off frequency (a>b)

$f_{20} \rightarrow TE_{20} \rightarrow$ second lowest cut off frequency

*In commercial waveguides (a=2b)

*In TM mode m= 0 n = 0 is not possible

The design of rectangular waveguides for a given frequency.

$$\frac{\lambda \in}{2} < a < \lambda \in \qquad \lambda \in = \frac{U}{f}$$

f= 1.6 Hz$\Rightarrow$ $\lambda \in = \dfrac{3.10^8 = 0{,}3m = 30cm}{10^9} = 0{,}3m = 30cm$

15 cm < a < 30 cm

THE WIDTHWISE EM COMPONENTS FOR TE_{mn} AND TM_{mn} MODES

Ht $= \dfrac{j\beta}{k} \nabla tHz = \beta / kc\left(\dfrac{\partial hz}{\partial x} az + \dfrac{\partial hz}{\partial y} ay \right) e^{-j\beta z}$

Ug= $(d\beta/_{dw})^{-1}$

Et= $-j\beta/_{kc}{}^{2}$ ∇t Ez

PROPAGATION SPECIALITIES

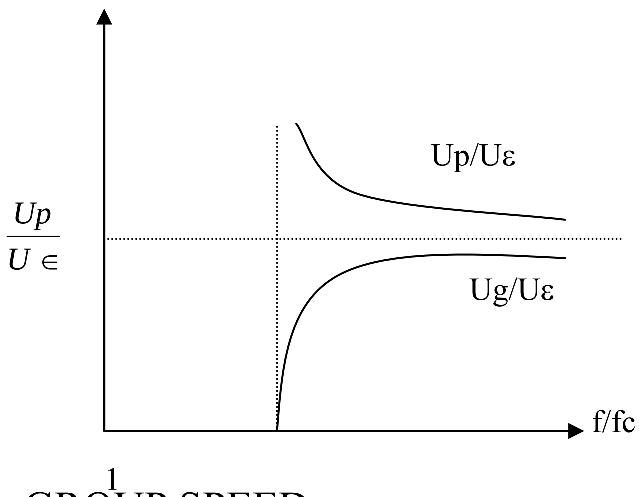
$$\beta = \frac{2\pi f}{U \in}\sqrt{1-\left(\frac{fmn}{f}\right)^2} \rightarrow \beta = \sqrt{k^2 - kmn^2} =$$

wt-βz=k the speed of constant phase lane

$$Up = \frac{w}{\beta} = \frac{U \in}{\sqrt{1-\left(\frac{fmn}{f}\right)^2}} > U \in$$

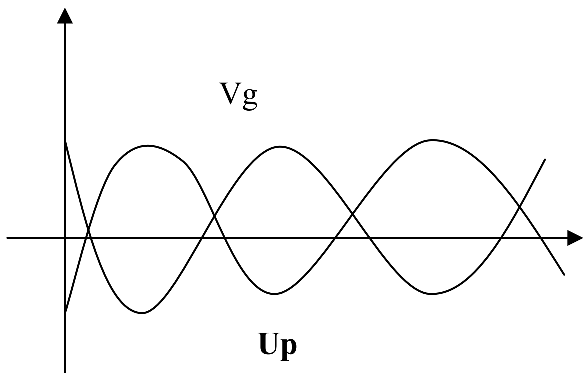
For general rectangular waveguide the speed of waves are bigger than the speed in space

$F_{mn} \rightarrow$ the cutoff frequency for TE_{mn} or TM_{mn}



GROUP SPEED

Ug= $1/_{d\beta/dw}$ = Uε $\sqrt{1-\left(fmn/f\right)^2} < U \in$



Ug.Up= $U\varepsilon^2$ Up> Ug

POWER

$$\vec{P}_{ort} = \frac{1}{2}\operatorname{Re}\left\{\vec{E}x\vec{H}^{\,*}\right\} w/_{m^2}$$

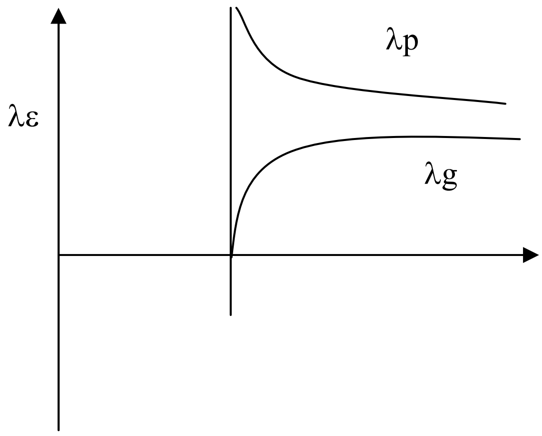
$$\vec{P}_{ort} = W_{ort}\,\vec{U} \in \Rightarrow U \in = \frac{PORT}{WORT} = Ug$$

Guided wave lenght

$$\lambda g = \frac{2\pi}{\beta} = \frac{\lambda \in}{\sqrt{1-\left(fmn/_f\right)^2}} > \lambda \in$$

$$\lambda \in = \frac{2\pi}{k} = \lambda g.\beta = 2\pi$$

The guided waves wave lenght decreases.

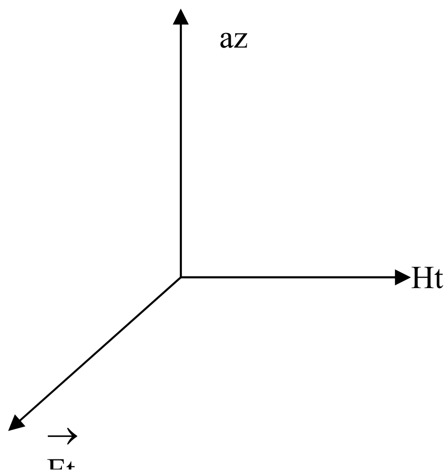


WAVE IMPEDANCES Z_{TE} , Z_{TM}

$$Z_{TEMN} = \frac{2}{\sqrt{1-\left(\frac{fmn}{f}\right)^2}} > \eta \quad \eta = \frac{377}{\sqrt{\in r}}$$

$$Z_{TM} = \eta\sqrt{1-\left(fmn/_f\right)^2} < \eta$$

$$Z_{\substack{T\in \\ TM}} = \frac{Et}{Ht} \qquad \eta = \frac{E}{H}$$



az:The direction of EM power propagation

CALCULATION OF P_{mn} (For TE_{mn} and TM_{mn})

$$Pmn = \iint_{widthwise} \vec{P}_{ort} \vec{d}s \rightarrow \text{ The net power propagating in the z direction}$$

$$Pmn = \frac{1}{2}\text{Re} \int_{x=0}^{a} \int_{y=0}^{b} \left(\vec{E}_t x \vec{H}^*{}_t\right) . \vec{a}z \, dx.dy$$

$$= \frac{1}{2}\text{Re} \int_{0}^{a} \int_{0}^{b} \left[ExH^* y - Ey.H^* x\right] dx.dy$$

$$= \frac{1}{2}\text{Re} \int_{0}^{a} \int_{0}^{b} \left[Hy.H^* y + Hx.Hx^*\right] dx.dy$$

$$Zwmn = \frac{Exmn}{Hymn} = \frac{Eymn}{Hxmn}$$

$$Pmn = \frac{1}{2}\text{Re}\, Zwmn \int_{widthwise\,section} \int | Hx |^2 + | Hy |^2) dxdy$$

By using H_x and Hy

$$Pmn = \frac{1}{2}\operatorname{Re} Zwmn \int_0^a \int_0^b (\sin^2 \frac{m\pi}{a} x + \cos^2 \frac{n\pi y}{b} dxdy$$

$$\begin{cases} \frac{ab^2}{4} & n \neq 0 \qquad m \neq 0 \\ \frac{ab}{2} & n = 0 \qquad m \neq 0 \end{cases}$$

$$\text{Zwmn } |\text{Ht}|^2 = \text{Zwmn } \frac{|Et|^2}{Zwn^2} = \frac{|Et|^2}{Zwn}$$

TOTAL EM POWER FOR TE$_{mn}$ or TM$_{mn}$ MODES

$$Pmn = \frac{|\,Hmn\,|^2\ ab}{2.\in on.\in om}$$

Here εom and εon are NEUMAN FACTORS

$$\in om = \begin{cases} 1 & m = 0 \\ 2 & m > 0 \end{cases}$$

$$\in on = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases}$$

$$\beta_{10} = \sqrt{k^2 - k_{10}{}^2} \quad k_{10} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

FOR Tε$_{10}$ MODE

$$P_{10}=\frac{1}{4}w\mu\beta_{10}\left(\frac{a}{\pi}\right)^{2}ab\,|H_{10}|^{2}$$

$$P_{10}=\frac{1}{4}\frac{1}{ZT\in_{10}}ab\,|E\max|^{2}$$

$$Emo\,x=\frac{w\mu oa\,Ho}{\pi}$$

$TE_{10} \rightarrow E_T = E_y\,ay$ Total field is only in the y direction.

M=1 n=0

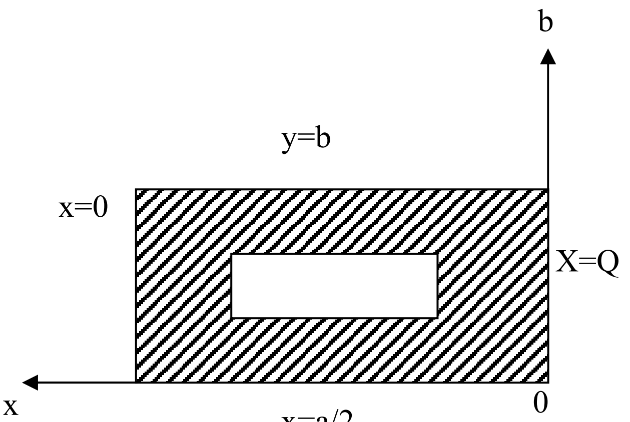
(DOMINANT MODE**)**

Emox < Edielectric distortion

From TE_{10} mode Ez=0   Hz≠0   Ex=0 can be find

E is only at y direction and at x= a/2 tthere is maximum Ey

The electrical fields is maximum at x=a/2. In other regions the change is sinπ/.



Ey= -jw $\mu_0 \frac{a}{\pi} H_{10} \sin\frac{\pi}{a} x e^{-j\beta z}$

Hx= jβ $\frac{a}{\pi} H_{10} \sin \frac{\pi}{a} x e^{-j\beta z}$

Hz= $H_{10} \cos\frac{\pi x}{a} e^{-j\beta z}$

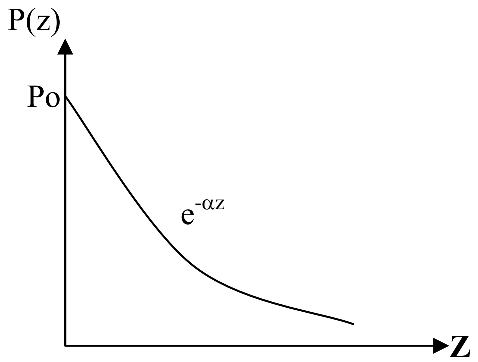
$$P_{10} = \frac{1}{4}\frac{1}{ZTE_{10}} a.b.E\max^2$$

Emax < Edielectric distortion happens

Emax ≥ Edielectrik distortion doesnot happens

If system is given then Emax can be find and maximum power occurs.

THE CONDUCTİVİTY LOSSES



$$P = \frac{1}{2Zw}\int Et\, H^{*}\, t\, ds = \frac{Zw}{2}\int Ht.Ht^{*}.\vec{d}s \qquad Ht\,\frac{1}{Zw}(azxEt)$$

$$\frac{Et}{Ht} = Zw$$

$$-\frac{\partial p}{\partial z} = PL = 2\alpha\, Poe^{-2\alpha z}$$

=2αP= 2(αc+αd)P

canductivity dielectric

lass loss

$$\alpha c = \frac{PL}{2P}$$

$$P_L = \frac{Rs}{2}\oint Ht.Ht^{*}\, dl$$

$Rs = \dfrac{1}{\delta gs}$ because of peffective depth there is a R_s surface impedance

$$\alpha c = \frac{Rs \oint_e Ht.Ht\,dl}{Zw \oint_s Ht.Ht\,ds}$$ (NP/M) conductivity loss constant

$\alpha c \rightarrow$ It is the result of ideal material

DIELECTRIC LOSSES

Eef = ε-j$\frac{\sqrt{d}}{w}$ d: dielectric conductivity

REMEMBER

$\nabla x \vec{H} = (\delta d + jw\varepsilon)\vec{E}$ $J\vec{u} = 0$

$\nabla x \vec{H} = jv(\varepsilon - j\frac{\delta d}{w})\vec{E}$

$\gamma = \alpha d + j\beta = j\sqrt{k^2 - kc^2}$

$\gamma = j\sqrt{w^2 \mu o \in ef - kc^2}$

$\gamma = \alpha d + j\beta$ and $w\mu o\ \sigma d <<< w^2\mu o\varepsilon - kc^2/$ and also with the use of binomial serials.

$$\alpha d = \frac{\sigma d}{2}\sqrt{\frac{\mu o}{\in}} \quad \left\{1-\left(\frac{wc}{w}\right)\right\}^{1/2} \quad Np/m\, w >> wc$$

The loosing factor in $e^{-\alpha dz}$, αd is real and positif.

The relationship bteween $\alpha B/_m$ and $Np/_m$ is

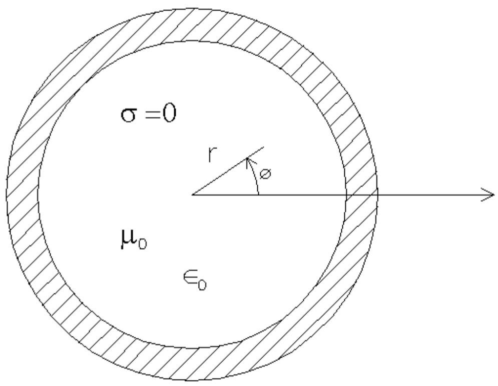
$$DB/m= \frac{10}{\Delta z}\log_{10} e^{2\alpha\Delta z} = 8,686\,\alpha$$

$$= \frac{10}{\Delta Z}\left[\log e^{2\alpha\Delta Z}\right] = 20\,x\,\alpha\,x\,Log_{10^e}$$

w>> wc için $\alpha d = \frac{\sqrt{d}}{2}\sqrt{\mu o/_{\in}}$

There are two losses. The αc is because of material not being ideal. The other loss becomes from cutoff frequency.

CIRCULAR WAVEGUIDE



The Circular Cylindrical Waveguide

This figure illustrates a cylindrical wave guide with a circular cross section of radius r. In view of the cylindrical geometry involved, cylindrical coordinates are most appropriate for the analysis to be carried out. Since the general properties of the modes that may exist are similar to those for the rectangular guide.

$$\nabla^2\psi + k_{\in}\psi = 0$$ Helmholtz Equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}+\left(k_{\in}-\beta^2\right)\psi = 0$$

(in the circular cylindrical coordinate)

$$e^{-j\beta z}\psi(r,\phi,z) = R(r)\,\Phi(\phi)e^{-j\beta z}$$

$$\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + h^2r^2}_{\text{only function of } r} = \underbrace{-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{only function of } \phi}$$

The left-hand side is a function of r only, whereas the right-hand side depend on ϕ only. Therefore this equation can hold for all values of the variables only if both sides are equal to some constant k^2.

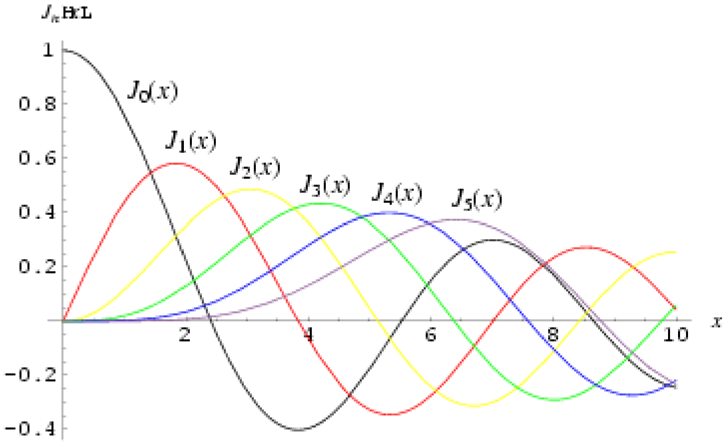
$$\frac{d^2\Phi}{d\phi^2}+\kappa^2\Phi = 0 \quad \Rightarrow \quad \Phi(\phi) = A\cos\kappa\phi + B\sin\kappa\phi \equiv C\cos(\kappa\phi+\varphi)$$

For given r, ϕ and $2n\pi+\phi$ represent same point.

For $\kappa = n$ n=0,1,2,….And $\phi = 0$. w

$$\Phi(\phi) = C\cos n\phi$$

$$r^2 \frac{d}{dr}\left(r \frac{dR}{dr} \right) + (h^2 r^2 - n^2)R = 0$$



(The Bessel Differential Equation)

$$\underbrace{DJ_n(hr)}_{\text{Bessel Function}} + \underbrace{EN_n(hr)}_{\text{Neumann Function}} = 0$$

Bessel Function   Neumann Function

In order to the function goes to infinite ,it should be E=0

$$\Psi(r,\phi,z) = DJ_n(hr)\cos n\phi \quad e^{-j\beta z} \quad ; \quad \beta^2 = k_\in{}^2 - h^2$$

Ψ (TE)$\rightarrow$H$_z$

$$\left.\frac{\partial \mathrm{H}_z}{\partial n}\right|_{equation \quad of \quad boundary} \equiv \left.\frac{\partial \mathrm{H}_z}{\partial r}\right|_{r=b} = 0$$

$E_z=0$; $J_n(hb)=0 \Rightarrow p_{nm} \Rightarrow q_{nm}$

TE_{11} , TM_{01} , TE_{21} , TE_{01}/TM_{11}

TE:

$J'_n(hb)=0 \rightarrow J'_n(q_{nm})=0 \rightarrow q_{nm}=hb$

$$f_{TE_{nm}}=\frac{q_{nm}U_{\in}}{2\pi b} \rightarrow h=\frac{q_{nm}}{b}=\frac{\omega_C}{U_{\in}}$$

$J_n(hb)=0 \rightarrow J_n(p_{nm})=0 \rightarrow p_{nm}=hb$

$$f_{TM_{nm}}=\frac{p_{nm}U_{\in}}{2\pi b}$$

TE and TM cutoff frequencies are different from each other.

Order of the modes w.r.t the cutoff frequencies (from low to high) ($\beta_{nm}=0$)

TE_{11} , TM_{01} , TE_{21} , TE_{01}/TM_{11}……..TE_{31}

EXAMPLE:

(a) f=6 GHz , 500 kW continuous wave power l=30 feet, choose a traditional(commercial available) circular wave guide,
(b) Order the lowest five cutoff frequencies,
(c) Find out the operation bandwidth for the TE_{11} mode,
(d) Find out the loss,
(e) Find out the maximum wave for electrical field strength And compare it with break down value for the dry air,
(f) If you insert a Teflon disk in the wave guide , in order to have it as invisible what should its thickness be?

SOLUTION:

(a)f=6 GHz ; $f_{c\,TE_{11}}=\frac{q_{11}U_{\in}}{2\pi b}$

The operation frequency has to be higher than $f_{c\,TE_{11}}$ for the safety margin let us choose.

$$f = 1.25x\ f_{c\,TE_{11}} \Rightarrow f_{c\,TE_{11}} = \frac{f}{1.25}$$

$$f_{c\,TE_{11}} < f < f_{c\,TE_2} \Rightarrow 1.25x\ f_{c\,TE_{11}} \leq f \leq 0.9x f_{c\,TE_2}$$

Taking

$$f_{c\,TE_{11}} = 5\ \text{GHz} \rightarrow f_{c\,TE_{11}} = \frac{1{,}841xc}{2\pi b} \Rightarrow 2b = 3{,}5\text{cm} = 1{,}39'' \Rightarrow \text{WC } 150$$

We choose standard WC 150 from the table of standard circular wave guides.

$$\text{WC } 150 \Rightarrow 2b = 1{,}5''$$

Wave guide⌋ ⌊Circular

for this value(2b=1,5'') we obtain: $f_{c\,TE_{11}} = 4{,}614$ GHz

EAI Designation	Inside Dimensions(Inches)			Recommended Frequency Range TE11 Mode GHz
	Diameter	Tolerance + or -	Roundness Tolerance	
WC 992	9,915	0,01	0,01	0,803-1,10
WC 847	8,47	0,008	0,008	0,939-1,29
WC 724	7,235	0,007	0,007	1,10-1,51
WC 618	6,181	0,006	0,006	1,29-1,76
WC 528	5,28	0,005	0,005	1,51-2,07
WC 451	4,511	0,005	0,005	1,76-2,42
WC 385	3,853	0,004	0,005	2,07-2,83
WC 329	3,292	0,003	0,003	2,42-3,31
WC 281	2,812	0,003	0,003	2,83-3,88
WC 240	2,403	0,0025	0,002	3,31-4,54
WC 205	2,047	0,002	0,002	3,89-5,33
WC 175	1,75	0,0015	0,0015	4,54-6,23
WC 150	1,5	0,0015	0,0015	5,30-7,27
WC 128	1,281	0,0013	0,0013	6,21-8,51
WC 109	1,094	0,001	0,0011	7,27-9,97
WC 94	0,938	0,0009	0,0009	8,49-11,6

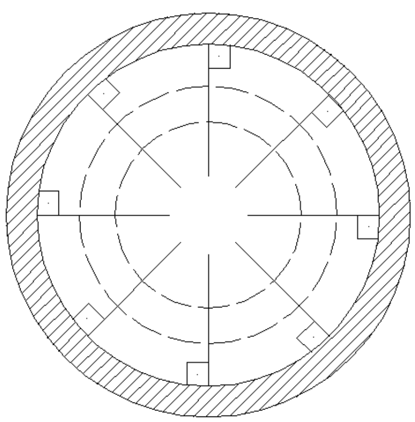
WC 80	0,797	0,0008	0,0008	9,97-13,7
WC 69	0,688	0,0007	0,0007	11,6-15,9
WC 59	0,594	0,0006	0,0006	13,4-18,4
WC 50	0,5	0,0005	0,0005	15,9-21,8
WC 44	0,438	0,00045	0,0004	18,2-24,9
WC 38	0,375	0,00038	0,0004	21,2-29,1
WC 33	0,328	0,00033	0,0003	24,3-33,2
WC 28	0,281	0,00028	0,0001	28,3-38,8
WC 25	0,25	0,00025	0,0001	31,8-43,6
WC 22	0,219	0,00025	0,0001	36,4-49,8
WC 19	0,188	0,00025	0,00007	42,4-58,1
WC 17	0,172	0,00025	0,00007	46,3-63,5
WC 14	0,141	0,00025	0,00005	56,6-77,5
WC 132	0,125	0,00025	0,00005	63,5-87,2
WC 11	0,109	0,00025	0,00005	72,7-99,7
WC 9	0,094	0,00025	0,00005	84,8-116

(b) The lowest five cutoff frequencies the WC 150

Mode:	TE_{11}	TM_{01}	TE_{21}	TE_{01}/TM_{11}
f_c(GHz):	4.614	6.028	7.654	9.604

(c) The operation bandwidth,

$$1.15\text{x}f_{c_{TE_{11}}} \le f \le 0.95\text{x}f_{c_{TE_{21}}}$$



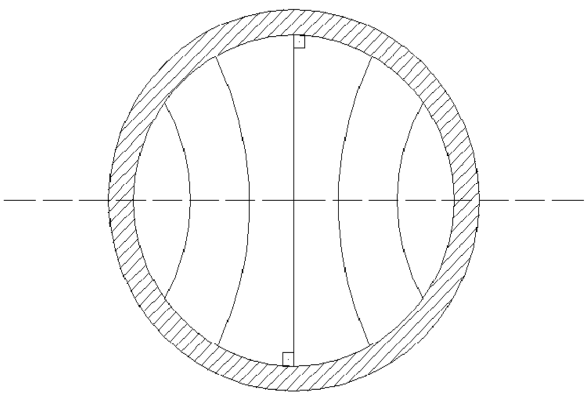
TM_{01}:
→ E_ lines

---- H_ lines

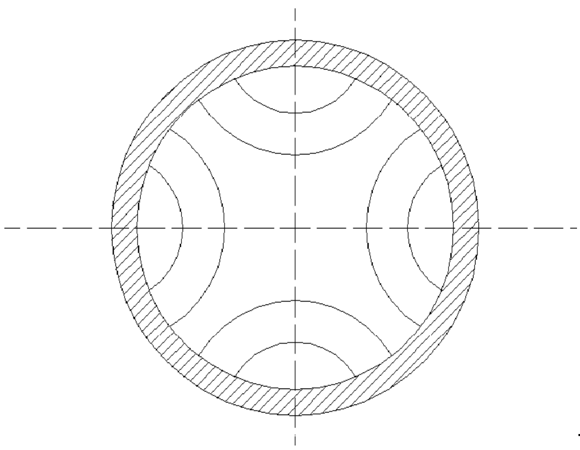
TM_{01} is not generally used for the second order mode, since this configuration does occur rarely in practice.

TE_{11}:

$\rightarrow$ E_ lines



TE_{21}:



$\rightarrow$ E_ lines

(d) $\alpha_{c\,TE_{nm}}$

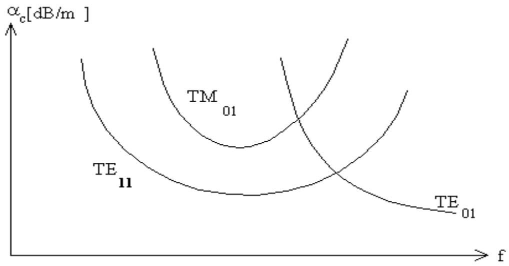
n: the order of the Bessel function , m: the order of the zeros

$$\alpha_{c\ TE_{nm}} = \frac{8.686}{\sigma\delta \quad b\zeta \quad \sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

For the 30'' propagation distance of 'Al' waveguide the loss power= 0.68 dB

$$\frac{outputpower}{inputpower} = \%85.4 \quad , \qquad P_{LOSS}=72.6 \text{ kW}$$

If the operation frequency f increases, the variations α_c as dB/m are given below:



For the atmosphere pressure, the circular wave guide with the dry air insulator, the maximum pulsive power can be

$$P_{\max}\Big|_{TE_{11}} = 2.7(2\mathrm{B})^2 \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$P_{\max}\Big|_{TE_{11}} = 3.88 \text{ MW} \qquad ;$$

$$E_{max} = \sqrt{\tfrac{0.5}{3.88}\,x29kV} = 10405 \text{ W/cm}$$